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FOURIER TRANSFORMS OF HOLOMORPHIC FUNCTIONS AND APPLICATION TO NEWTON INTERPOLATION SERIES, I

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Fourier transforms of holomorphic functions and application to Newton interpolation series I

J.W. de Roever

ABSTRACT

In this paper we consider spaces of analytic functions. Our main tool is the theory of Fourier transformation for generalized functions. The spaces of analytic functions we are interested in are Fourier transforms of some spaces of generalized functions. The analytic functions don't need to be entire, but they are holomorphic in tubular radial domains. The relation between the support of a generalized function and the exponential growth at infinity of its Fourier transform is pointed out. We use this to derive the Newton interpolation series for a wider class of functions. For that purpose we introduce the dual space of the space of generalized functions.

A subsequent paper (part II) will deal with analytic functions being Fourier transforms of analytic functionals.

CONTENTS

1. INTRODUCTION

In [3] Kioustelidis derives the Newton interpolation series for entire functions by means of Fourier transformation. E.M. de Jager suggested that using a Paley-Wiener-Schwartz type theorem one would apply the same method to non-entire functions. In this paper this suggestion is worked out. Further, attention is paid to the spaces of holomorphic functions, because they are important in partial differential equations related to quantum physics. For that purpose the above-mentioned theorem, that can be found in [11], is generalized somewhat, while a more fundamental generalization will appear in part II.

First we define the Newton interpolation series. Let f be an entire function; for any vector $h \in \mathbb{C}^n$ Taylor expansion yields

$$f(z+h) = \sum_{k=0}^{\infty} \frac{(h\cdot D)^k}{k!} f(z) \overset{\text{def}}{=\!=} \exp (h\cdot D)\, f(z)$$

with D the vector $(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n})$. For $s \in \mathbb{C}$ we can formally write

$$\begin{aligned} f(z+ish) &= \exp (ish\cdot D)\, f(z) = (\exp ih\cdot D)^s f(z) = \\ &= (1+\Delta_{ih})^s f(z) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta_{ih}^k f(z) \qquad (\textit{Newton series}), \end{aligned} \tag{1.1}$$

where $\Delta_{ih} \overset{\text{def}}{=\!=} \exp (ih\cdot D) - 1$, so that $\Delta_{ih} f(z) = f(z+ih) - f(z)$ and

$$\Delta_{ih}^k f(z) = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} f(z+imh). \tag{1.2}$$

The polynomials $\binom{s}{k} = s(s-1)\cdots(s-k+1)/k!$ are the Newton polynomials $p_k(s)$. Usually the factor i is deleted in (1.1), but in this paper it will appear to be convenient to use formula (1.1) for the Newton series.

The Newton interpolation series (1.1) has been studied for example in [1] and [4]. In [1] the series has been derived for entire functions in one complex variable, while in [4] these functions are holomorphic in a half-space only. Recently formula (1.1) has been proved rigorously for entire functions in several complex variables of exponential growth by Kioustelidis in [3]. In this paper we will use the same method as Kioustelidis, namely Fourier transformation. In our case, however, the holomorphic functions don't need to be entire, it is sufficient that they are holomorphic in tubular domains.

In order to prove formula (1.1) we apply Fourier transformation to the formal operator exp $ih \cdot D$ and obtain the function $e^{-h\zeta}$. We restrict ourselves to functions f having a Fourier transform g with support K such that the series

$$(1.3) \qquad \sum_{k=0}^{\infty} \binom{s}{k} (e^{-h\zeta}-1)^k$$

converges for every $\zeta \in K$. K may be a real subset of $\mathbb{R}^n$ or a complex subset of $\mathbb{C}^n$. In the first case, which will be treated in this paper (part I), g is a generalized function and we will write ξ instead of ζ. In the second case, which will be treated in part II, g is an analytical functional carried by K.

We will give several spaces of functions holomorphic in tubular radial domains and the spaces of their Fourier transforms. They are topologized in such a way that Fourier transform is a topological isomorphism. Moreover, we will consider the dual spaces of the spaces of Fourier transforms; it is in these dual spaces that the series (1.3) should converge. We will determine the conditions on which the Newton series is valid for holomorphic functions belonging to one of these spaces and we will give the topology in which the series converges. In some of the spaces one may define the distributional boundary value on the distinguished boundary of the domain of holomorphy and it turns out that the Newton series is valid in distributional sense on the distinguished boundary. Several properties of the spaces, like nuclearity, are mentioned. These properties are not strictly needed for the proof of the Newton series, but they may be important for other applications, like tensor products and representations of kernels.

Section 2 contains a survey of notations, definitions and preliminary theorems. Sections 3 and 4 are devoted to the well-known case of entire functions of polynomial growth in $|\mathrm{Re}\, z|$ and of exponential growth in $|\mathrm{Im}\, z|$. This is a special case of entire functions of exponential growth in $|z|$, which will be considered in part II and which has been treated by Kioustelidis in [3]. However, we get a slightly stronger result on the convergence of the series (1.1). In section 4 formula (1.1) is derived with the aid of the theorem of Paley-Wiener-Schwartz, which is treated in section 3. In section 5 we give some properties of the series (1.3) and discuss possible generalizations of the results of sections 3 and 4. The exponential growth at infinity of a holomorphic function f determines the support of its Fourier transform g, which is compact as long as f is entire. Sections 6, 8 and 9 deal with functions f holomorphic in tubular radial domains. Now the support of g is no longer compact. In section 6 f has a distributional boundary value in S', the space of tempered distributions. Furthermore section 6 describes the topologies of the spaces of such functions and of its Fourier transforms g. We will also consider the dual space of the last one. In section 7 the Newton series (1.1) is derived for the functions f of section 6. The holomorphic functions f of section 8 only have limits to the distinguished boundary in Z', the Fourier transform of the space D' of Schwartz-distributions. Moreover, section 8 describes the Fourier transforms g of such functions f. In section 9 the topologies of the spaces of the functions f and distributions g are given. Finally section 10 treats the Newton series for these functions f.

2. NOTATIONS AND PRELIMINARIES

A.1. We denote by $\mathbb{R}^n$ the real and by $\mathbb{C}^n$ the complex n-dimensional vector space; we denote the components of $z \in \mathbb{C}^n$ by z_j and write $z = x + iy$ with x and y in $\mathbb{R}^n$; dx dy means $dx_1 \dots dx_n dy_1 \dots dy_n$. For $u = (u_1,\dots,u_n)$ and $v = (v_1,\dots,v_n)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ we mean by $u \cdot v$ the product $u_1v_1 + \dots + u_nv_n$; the product of a vector u in $\mathbb{R}^n$ or $\mathbb{C}^n$ by a scalar λ is denoted as λu. The norm of a vector u in $\mathbb{R}^n$ and z in $\mathbb{C}^n$ is $\|u\| = \sqrt{u \cdot u}$ and $\|z\| = \sqrt{z \cdot \bar{z}}$. If $u \in \mathbb{R}^n$, $\tilde{u}$ will be the unit vector $u/\|u\|$ in the direction of u. With α a multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ we write u^α for $u_1^{\alpha_1} \ldots u_n^{\alpha_n}$,

D^α or D_x^α for $\dfrac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $|\alpha|$ for $\alpha_1+\ldots+\alpha_n$ and $\alpha!$ for $\alpha_1! \ldots \alpha_n!$;

by u^2 we mean $u \cdot u$. If c is a real constant, then $\bar{c}$ is the vector $(c,\ldots,c)$ in $\mathbb{R}^n$.

A.2. If $V \subset \mathbb{R}^n$ or $\mathbb{C}^n$ is an open set, we denote its *closure* by $\bar{V}$, its *complement* by V^c, its *boundary* by ∂V and its *convex envelope* by $O(V)$. If V is *relatively compact* we write $V \subset\subset \mathbb{R}^n$. By T^B we mean the tubular domain $\mathbb{R}^n + iB$ in $\mathbb{C}^n$.

A.3. A *cone* C in $\mathbb{R}^n$ is defined by requiring that $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$. Each cone C defines a set pr C, the *projection* of C, on the unit sphere S in $\mathbb{R}^n$ by pr $C = \{y \mid y \in C \text{ and } \|y\| = 1\}$. On S we take the topology induced by $\mathbb{R}^n$ and the topology of pr C is determined as a subset of S. It is clear that a cone $C \neq \mathbb{R}^n$ is open if and only if pr C is open and $0 \notin C$. The *dual cone* C^* of a cone C is defined by $C^* = \{\xi \mid y \cdot \xi \geq 0 \text{ for all } y \in C\}$. We call a cone C' a *relatively compact subcone* of the open cone C, denoted by $C' \subset\subset C$, if $\overline{\text{pr } C'} \subset \text{pr } C$.

A.4. In the sequel C will be an open convex cone in $\mathbb{R}^n$ and $\{C_k\}_{k=1}^\infty$ a sequence of open relatively compact subcones of C with $C_k \subset\subset C_{k+1} \subset\subset C$ for $k=1,2,\ldots$ and $\bigcup_{k=1}^\infty C_k = C$. We denote by α_k the minimum distance of $\partial \text{pr } C_k$ to $\partial \text{pr } C_{k+1}$ in radials and $\delta_k = \sin \alpha_k$.

A.5 If $a(\tilde{y})$ is a continuous function on pr C, we will denote by $\tilde{a}(y)$ the function $\|y\|\, a(\tilde{y})$, which is homogeneous of degree 1 on C. When this last function $\tilde{a}$ is convex, that means, for all $y_1, y_2 \in C$ and $0 \leq \lambda \leq 1$, $\tilde{a}(\lambda y_1 + (1-\lambda) y_2) \leq \lambda\, \tilde{a}(y_1) + (1-\lambda)\, \tilde{a}(y_2)$, we also call the function a on pr C *convex*.

A.6. Let E be a topological vector space, then E' will be the strong deal and $\langle .,. \rangle$ the bilinear form defining the duality between E and E'.

B. Now we give some definitions of function spaces, which can be found in Wloka [12], except the spaces in B.2 and B.3. Let $O \subset \mathbb{R}^n$ be open and M a positive continuous weight function on O.

B.1. The space $W_\infty^m(M,O)$ is defined as the set of C^m functions ϕ on O for which

$$\|\phi\|_m \overset{\text{def}}{=} \sup_{\substack{\xi\in O\\ |\alpha|\le m}} M(\xi)\,|D^\alpha\phi(\xi)| < \infty.$$

With $\|\cdot\|_m$ as norm $W_\infty^m(M,O)$ is a Banach space; we also write $\|\cdot\|_{m,O}$ for the norm. Sometimes M is labeled: $M = M_p$ and we denote the norm by $\|\cdot\|_m^p$ or if O is labeled too: $O = O_1$, by $\|\cdot\|_{m,1}$ or $\|\cdot\|_{m,1}^p$. If $p = 1 = m$ we just write $\|\cdot\|_{m,m} = \|\cdot\|_{m,m}^m = \|\cdot\|_m$.

B.2. Now let M be continuous and positive on $\bar{O}$. Then we define $W_\infty^m(M,\bar{O})$ as the subspace of $W_\infty^m(M,O)$, for which $\phi \in W_\infty^m(M,O)$ and its derivatives $D^\alpha\phi$ on O can be extended to continuous functions on $\bar{O}$. $W_\infty^m(M,\bar{O})$ is a closed linear subspace of $W_\infty^m(M,O)$, so it is a Banach space with the norm given in B.1.

B.3. The subspace $W_{\infty,0}^m(M,\bar{O})$ of $W_\infty^m(M,\bar{O})$ is defined in case $\bar{O}$ is not compact by the requirement that

$$\lim_{\xi\to\infty,\,\xi\in\bar{O}} M(\xi)D^\alpha\phi(\xi) = 0 \qquad \text{for } |\alpha| \le m.$$

This subspace is closed, so that $W_{\infty,0}^m(M,\bar{O})$ is a Banach space with the norm given in B.1.

B.4. Let now $\Omega \subset \mathbb{C}^n$ be open and M a positive continuous weight function on Ω. We define $A_\infty(M,\Omega)$ as the set of functions holomorphic in Ω for which

$$\|f\| = \sup_{z\in\Omega} M(z)\,|f(z)| < \infty.$$

With this norm $A_\infty(M,\Omega)$ is a Banach space. If M is labeled: $M = M_m$ we denote this norm by $\|\cdot\|^m$ and if moreover $\Omega = \Omega_k$, by $\|\cdot\|_k^m$.

B.5. We can identify $W^m_{\infty,0}(M,\bar{O})$ with a closed linear subspace of $\Pi_{|\alpha|\le m} C^0(\bar{O})$ by means of the map $\phi \to (MD^\alpha\phi)_{|\alpha|\le m}$. With the aid of the theorem of Hahn-Banach we can describe the continuous linear functionals f on the space $W^m_{\infty,0}(M,\bar{O})$:

$$\phi \in W^m_{\infty,0}(M,\bar{O}): \quad \langle f,\phi\rangle = \sum_{|\alpha|\le m} \int_{\bar{O}} M(\xi)\, D^\alpha\phi(\xi)\, d\mu_\alpha(\xi)$$

with μ_α bounded Radon measures in $\bar{O}$ for $|\alpha| \le m$ and

$$\|f\| = \sum_{|\alpha|\le m} \int_{\bar{O}} |d\mu_\alpha(\xi)|$$

see [7] V, §18.5.5. The norm dual to the norm of $W^m_{\infty,0}(M,\bar{O})$ is

$$\|f\|_{-m} = \inf_{f=(g_\alpha)} \|g\|$$

where the infimum is computed over all representations of the form $f = (g_\alpha)_{|\alpha|\le m}$ with $g_\alpha \in (C^0(\bar{O}))'$.

C. In what follows we discuss some properties of the identity map between several W- and A-spaces, which again can be found in Wloka [12], except C.2. Strictly, the identity maps are restrictions from functions on a large domain to a smaller domain and they are not necessarily injective. Let $O_2 \subset O_1 \subset \mathbb{R}^n$ be open sets, $m \le 1$ and $M_1(\xi) \ge M_2(\xi)$ for $\xi \in O_2$.

C.1. The identity map: $W^m_\infty(M_1,O_1) \to W^1_\infty(M_2,O_2)$ is continuous.

C.2 If, morevoer, $\bar{O}_2 \subset O_1$, M_1 is continuous and positive on $\bar{O}_1$ and M_2 on $\bar{O}_2$, and

$$\lim_{\xi\to\infty,\, \xi\in\bar{O}_2} M_2(\xi)/M_1(\xi) = 0 ,$$

then the following identity maps are continuous:

$$W^m_\infty(M_1,O_1) \to W^m_\infty(M_1,\bar{O}_2) \to W^m_{\infty,0}(M_2,\bar{O}_2) \to W^m_\infty(M_2,\bar{O}_2) \to W^m_\infty(M_2,O_2).$$

C.3. Similarly to C.1 continuous identity maps exist between A-spaces; here they are embedding maps, that is injective, for a holomorphic function is determined by its values on an open set in $\mathbb{C}^n$. Let $\Omega_2 \subset \Omega_1 \subset \mathbb{C}^n$ be open sets and $M_1(z) \geq M_2(z)$ for $z \in \Omega_2$.

The embedding $A_\infty(M_1,\Omega_1) \subset\to A_\infty(M_2,\Omega_2)$ is continuous.

Let E and F be two locally convex Hausdorff spaces. A linear map from E into F is called compact if it transforms an open neighborhood of zero in E into a relatively compact set of F. If E is a Banach space, this means that it must transform every bounded set into a relatively compact set. We now give conditions in order that the identity maps in C.1 and C.3 are compact. We suppose $m > 1$.

C.4. If there are open relatively compact subsets K_k of O_2 with for $k=1,2,\ldots$ $\bar{K}_k \subset K_{k+1} \subset O_2$ and $\cup_k K_k = O_2$ and if for all $\varepsilon > 0$ there is a $k(\varepsilon)$ such that, for all $\xi \in O_2 \setminus K_{k(\varepsilon)}$, $M_2(\xi) \leq \varepsilon M_1(\xi)$, then the identity map from $W^m_\infty(M_1,O_1)$ into $W^1_\infty(M_2,O_2)$ is compact.

C.5. By the same reasoning as in [12] one can proof that in C.4 the condition $\bar{K}_k \subset K_{k+1}$ can be replaced by $\bar{K}_k \subset O_1$ for all $k=1,2,\ldots$.

C.6. If $\Omega_2 = \cup_k S_k$ with $\bar{S}_k \subset S_{k+1} \subset \mathbb{C}^n$, where the sets S_k are open relatively compact subsets of Ω_2 and if for all $\varepsilon > 0$ there is a $k(\varepsilon)$ such that, for all $z \in \Omega_2 \setminus S_{k(\varepsilon)}$, $M_2(z) \leq \varepsilon M_1(z)$, then the embedding from $A_\infty(M_1,\Omega_1)$ into $A_\infty(M_2,\Omega_2)$ is compact.

C.7. As in C.5 we can replace the condition $\bar{S}_k \subset S_{k+1}$ in C.6 by $\bar{S}_k \subset \Omega_1$ for all $k=1,2,\ldots$.

D. We consider a special case, namely when $M(\xi)$ is equal to

$$M_{a,b}(\xi) = (1+\|\xi\|)^a e^{b\|\xi\|},$$

where a and b are two real numbers, not necessarily positive.

D.1. Let F be a closed convex set in $\mathbb{R}^n$. Then according to Whitney [9] each function $\phi \in W^m_\infty(1;F)$ (see definition B.2) is a C^m-function on the closed set F. In that case ϕ can be extended to a C^m-function ψ

on $\mathbb{R}^n$, which is bounded together with its derivatives on an open neighborhood U of F, see Whitney [10]. Thus the restriction map from $W^m_\infty(1;U)$ into $W^m_\infty(1;F)$ is surjective. Then according to the open mapping theorem (see [8]) a constant $K > 0$ exists such that for all $\phi \in W^m_\infty(1;F)$, there is a $\psi \in W^m_\infty(1;U)$ with $D^\alpha\psi(\xi) = D^\alpha\phi(\xi)$ for $\xi \in F$, $|\alpha| \le m$, and with $\|\psi\|_{m,U} \le K \|\phi\|_{m,F}$. [1)]

When all $D^\alpha\phi$, α such that $|\alpha| = m$, are uniformly continuous on F (see [9]), we can take for U an arbitrary open ε-neighborhood U_ε of F, see [10]. [2)] In that case there exists a C^∞-function σ on $\mathbb{R}^n$, equal to 1 on F and to 0 outside U_ε, whose derivatives are bounded on $\mathbb{R}^n$. Multiplying ψ by σ we get an extension $\tilde{\phi} = \sigma\psi$ of ϕ, that is bounded on $\mathbb{R}^n$ just as its derivatives. For example a function $\phi \in W^m_{\infty,0}(1;F)$ and its derivatives $D^\alpha\phi$, $|\alpha| \le m$, are uniformly continuous on F. Furthermore the construction of ψ in [10] shows that also $D^\alpha\psi$ approaches zero as ξ tends to infinity inside U_ε, so that $\tilde{\phi}$ belongs to $W^m_{\infty,0}(1;\mathbb{R}^n)$. Thus the restriction map from $W^m_{\infty,0}(1;\mathbb{R}^n)$ into $W^m_{\infty,0}(1;F)$ is surjective and again it follows that there is a constant K such that

$$\|\tilde{\phi}\|_{m,\mathbb{R}^n} \le K\|\phi\|_{m,F}.$$

On $W^m_\infty(M_{a,b};U)$ the norm

$$\|\phi\|_m = \sup_{\xi\in U,\ |\alpha|\le m} (1+\|\xi\|)^a e^{b\|\xi\|}|D^\alpha\phi(\xi)|$$

is equivalent to the norm

$$\sup_{\xi\in U,\ |\alpha|\le m} |D^\alpha(1+\|\xi\|)^a e^{b\|\xi\|}\phi(\xi)|.$$

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1) See [6], p.98, for more general conditions on F, under which $\|\psi\|_{m-l,U} \le K\|\phi\|_{m,F}$ for some nonnegative $l \le m$ depending on F.

2) The uniform continuity is not necessary, see [11].

Also the assertion

$$\lim_{\xi\to\infty,\,\xi\in U} (1+\|\xi\|)^{a} e^{b\|\xi\|} D^{\alpha}\phi(\xi) = 0$$

for all α with $|\alpha| \le m$ is equivalent to the assertion

$$\lim_{\xi\to\infty,\,\xi\in U} D^{\alpha}(1+\|\xi\|)^{a} e^{b\|\xi\|} \phi(\xi) = 0$$

for all α with $|\alpha| \le m$. Therefore we can conclude that the restriction map

$$I:\ W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n) \to W^{m}_{\infty,0}(M_{a,b};F)$$

is surjective, so that there is a K > 0 with

$$\|\tilde{\phi}\|_{m,\mathbb{R}^n} \le \|\phi\|_{m,F} \quad .$$ [3)]

Hence there is a continuous right-inverse map J of I, $J\phi = \tilde{\phi}$ and $I\circ J\phi = \phi$, from $W^{m}_{\infty,0}(M_{a,b};F)$ into $W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n)$.

D.2. From D.1 it follows that for any bounded set B in $W^{m}_{\infty,0}(M_{a,b};F)$ there is a bounded set B_1 in $W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n)$ with $IB_1 = B$, namely $B_1 = \{\chi \mid \chi = J\phi, \phi \in B\}$. Thus the image under the transpose map I' of the open set $V = \{f \mid |\langle f,\phi\rangle| < \varepsilon, \phi \in B\}$ in $(W^{m}_{\infty,0}(M_{a,b};F))'$ is $I'V = \{g \mid g \in \operatorname{Im} I', |\langle g,\chi\rangle| < \varepsilon, \chi \in B_1\}$, which is open in $(W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n))' \cap \operatorname{Im} I'$. Since I is surjective, I' is 1 - 1 and $(W^{m}_{\infty,0}(M_{a,b};F))'$ is a closed linear subspace of $(W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n))'$.

Remark. In fact the surjectivity of I implies that Im I' is weakly closed (see [8, theorem 37.2]) and from this it follows that $(W^{m}_{\infty,0}(M_{a,b};F))'$ can be identified (by means of I') with the subspace W'(F) of $(W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n))'$ consisting of the elements with support in F. Indeed, W'(F), which is defined as those f in $(W^{m}_{\infty,0}(M_{a,b};\mathbb{R}^n))'$ with

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3) See [11] for the space $W^{m}_{\infty}(M_{m,0};F)$; also there, more general conditions on F are given in order that $\|\tilde{\phi}\|_{m-1,\mathbb{R}^n} \le K\|\phi\|_{m,F}$.

$\langle f,\phi\rangle = 0$ for all ϕ in $W^m_{\infty,0}(M_{a,b};\mathbb{R}^n)$ having their support in F^c, also vanishes on the closure of the space of such ϕ, which is just Ker I. Then according to [8, prop. 35.4] $W'(F)$ is the weak closure of Im I'.

So we see from B.5 that the elements of $(W^m_{\infty,0}(M_{a,b};\mathbb{R}^n))'$ with support in F can be represented as derivatives of measures in F. This is not true for general sets F.

D.3. Let $\bar{O}_1$ be a convex closed set and O_2 a subset not necessarily open, contained in or equal to $\bar{O}_1$. Further, let $M_1 = M_{a,b}$ and let M_2 be a positive continuous weight function on O_2, such that for all $\varepsilon > 0$, there is a $R > 0$, so that, for all $\xi \in O_2$ with $\|\xi\| \geq R$, $M_2(\xi) \leq \varepsilon M_1(\xi)$. For example $M_2 = M_{c,d}$ with $c < a$ or $d < b$. Then according to C.5 the restriction map I from $W^m_{\infty,0}(M_1;\mathbb{R}^n)$ into $W^1_{\infty,0}(M_2;O_2)$ is compact $(m>1)$, because $W_{\infty,0}$-spaces are subspaces of W_∞-spaces. Hence also the restriction map $I\circ J$ from $W^m_{\infty,0}(M_1;\bar{O}_1)$ into $W^1_{\infty,0}(M_2;O_2)$ is compact. Thus for $\bar{O}_1$ closed and convex and $M_1 = M_{a,b}$ we have found a more general condition for the map in C.6 to be compact.

E. We will need the following two theorems about compact mappings; they can be found in [2].

E.1. (Schauders theorem): If the map $T: E \to F$ (E and F are normed spaces) is compact, then the transpose map $T': F' \to E'$ between the strong duals is compact too.

E.2. A compact map between normed spaces transforms weakly convergent sequences into convergent sequences.

Further we will need two theorems, one is the form of the Banach-Steinhaus theorem given in [11] and the other is Bochner's theorem, which can be found in [11] too.

E.3. If a sequence of linear continuous functionals f_n on a Banach space B with norm $\|\cdot\|_m$ has a weak limit on a dense subset of B and if the norms

$$\|f_n\|_{-m} = \sup_{\phi\in B} \frac{|\langle f_n,\phi\rangle|}{\|\phi\|_m} < C \text{ for all } n,$$

then the sequence converges weakly to a continuous linear functional on B.

E.4. If a function f is holomorphic in the tubular domain $T^B = \mathbb{R}^n + iB$ with B an open connected set in $\mathbb{R}^n$, where $n \geq 2$, then f is holomorphic in the convex envelope $O(T^B) = \mathbb{R}^n + iO(B)$ too.

F. We give some properties of countable projective and inductive limits, which can all be found in [2]. We only consider inductive limits that are composed of spaces injectively embedded in each other.

F.1. Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of locally convex Hausdorff spaces and $\pi_{mp}: E_m \to E_p$ be continuous maps. In case for each p there is a $m > p$ such that a map π_{mp} exists, the *projective limit* $E = \text{proj lim}_m E_m$ is defined as the subspace of $\Pi_1^{\infty} E_m$ for which $\pi_{mp} \circ \pi_m = \pi_p$, where π_p is the projection $\Pi_m E_m \to E_p$, provided with the relative product topology, that is the least fine topology such that all the maps $\pi_m: E \to E_m$ are continuous. In case for each m there is a $p > m$ such that π_{mp} exists and is injective, we define the *inductive limit* $E = \text{ind lim}_m E_m$ as $E = \cup_1^{\infty} E_m$ provided with the finest locally convex topology such that the embedding maps $\pi_m: E_m \to E$ are continuous; here we consider E_m as a subset of E_p so that $\pi_p \circ \pi_{mp} = \pi_m$.

F.2. The projective limit of complete spaces is itself complete and a projective limit of Banach spaces is a Fréchet space

F.3. Both, projective and inductive limits, are Hausdorff spaces.

F.4. A continuous map T from a locally convex space F into $\text{proj lim}_m E_m$ is continuous if and only if all the maps $\pi_m \circ T$ from F into E_m are continuous. A continuous map T from $\text{ind lim}_m E_m$ into a locally convex space F is continuous if and only if all the maps $T \circ \pi_m$ from E_m into F are continuous.

F.5. From F.4 follows: if $E = \text{proj lim}_m E_m$ and $F = \text{proj lim}_m F_m$ both are projective limits (if $E = \text{ind lim}_m E_m$ and $F = \text{ind lim}_m F_m$ both are inductive limits) and if for infinitely many m continuous maps $T_m: E_m \to F_{m'}$, exist, where $m' \leq m$ ($m' \geq m$) depends on m, such that

$T_p \circ \pi_{mp} = \pi_{m'p'} \circ T_m$, where $p < m$ and $p' < m'$ ($p>m$ and $p'>m'$), then the maps T_m induce a continuous map from E into F.

F.6. From F.5 follows: if $E = \text{proj lim}_m E_m$ ($E = \text{ind lim}_m E_m$) and if for all $l=1,2,\ldots$ locally convex spaces F_l and continuous maps $S_l: E_p \to F_l$ and $T_l: F_l \to E_m$ exist with $p \ge l \ge m$ ($p \le l \le m$), such that $\pi_{pm} = T_l \circ S_l$, then also $E = \text{proj lim}_l F_l$ ($E = \text{ind lim}_l F_l$).

F.7. From F.4 and F.5 follows: a projective limit $E = \text{proj lim}_m E_m$ of complete spaces E_m is equal to $E = \text{proj lim}_m E^{\bar{m}}$ where $E^{\bar{m}}$ is the closure of E in the topology of E_m.

F.8. If for all m the maps π_{pm} (π_{mp}) are compact for some $p > m$ depending on m, we call the projective limit an $F\bar{S}$-space (the inductive limit an LS-space).

F.9. $F\bar{S}$-spaces (LS-spaces) can be represented as projective limits (inductive limits) of Banach spaces, so that according to F.2 an $F\bar{S}$-space is a Fréchet space.

F.10. LS- and $F\bar{S}$-spaces are Montel spaces. Montel spaces are reflexive; the strong dual of a Montel space is again a Montel space. In a Montel space a bounded set is relatively compact and weakly convergent sequences in the dual are strongly convergent.

F.11. LS- and $F\bar{S}$-spaces are bornological. A bounded map from a bornological space into a locally convex space is always continuous.

F.12. The strong dual E' of an LS-space $E = \text{ind lim}_m E_m$ is the $F\bar{S}$-space $\text{proj lim}_m E'_m$. The strong dual E' of an $F\bar{S}$-space $E = \text{proj lim}_m E_m$, where $\pi_{mp} E_m$ is dense in E_p (then $\pi_m E$ is dense in E_m too; see also F.7), is the LS-space $\text{ind lim}_m E'_m$.

F.13. If E_m is embedded isomorphically into E_p by π_{mp} ($p>m$), that is if the topology of E_m is the one induced by E_p, we call the inductive limit *strict*.

F.14. LS-spaces and strict inductive limits of complete spaces are complete.

F.15. An inductive limit $E = \text{ind lim}_m E_m$ is called *regular*, if every bounded set is contained in some E_m and is bounded there.

F.16. LS-spaces and strict inductive limits of complete spaces are regular.

G. We apply the results of F to the W- and A-spaces of section B. We will pay special attention to being nuclear. We do not say when a space is nuclear (see for this [2] or [8]), but we merely give conditions in order that the occurring spaces are nuclear. Let $W(O) = \text{proj lim}_m W^m_\infty(M_m;O_m)$, with $O_{m+1} \supset O_m$ open in $\mathbb{R}^n$, $O = \cup_m O_m$ and $M_{m+1}(\xi) \geq M_m(\xi)$ for $\xi \in O_m$.

G.1. If the maps $\pi_{m+1,m}$ satisfy the conditions in C.4 or C.5 W(O) is an $F\bar{S}$-space. According to C.2 and F.6 the same conditions imply that W(O) can be represented as $\text{proj lim}_m W^m_{\infty,0}(M_m;O_m)$. In view of B.5 and F.12 the representation of W(O) as projective limit of $W_{\infty,0}$-spaces facilitates the description of the dual.

G.2. When W(O) is an $F\bar{S}$-space, a sequence ϕ_k converges in W(O), if $\phi_k(\xi)$ converges for each $\xi \in O$ and if $\|\phi_k\|_m \leq K_m$, where K_m are constants independent of k for m=1,2,... .

G.3. We will consider two conditions HS_1 and HS_2 (see [12]) on the weight functions M_m, which imply the conditions in C.4 or C.5. HS_1 and HS_2 make sure that W(O) is a nuclear $F\bar{S}$-space. An $F\bar{S}$-space is nuclear, if it can be represented as a projective limit of Hilbert spaces, where the maps π_{pm} are Hilbert-Schmidt type maps (see [12]). As Hilbert-spaces we take the spaces $W^m_2(M_m;O_m)$ of all measurable functions ϕ, for which the weak derivatives $D^\alpha\phi$ exist when $|\alpha| \leq m$, such that $MD^\alpha\phi$ belongs to $L^2(O_m)$. The inproduct is

$$(\phi,\psi) = \sum_{|\alpha|\leq m} \int_{O_m} M^2(\xi)\, D^\alpha\phi(\xi)\cdot D^\alpha\psi(\xi)d\xi.$$

Let HS_1 be: for all m there is a p > m, such that

$$(HS_1) \qquad \int_{O_m} \left(\frac{M_m(\xi)}{M_p(\xi)}\right)^2 d\xi < \infty .$$

This implies that the restriction map $S_{p,m}$ from $W^p_\infty(M_p;O_p)$ into $W^m_2(M_m;O_m)$ is continuous. Let HS_2 be: for all m there is a $p > m + n/2$ such that

(HS_2) O_m can be covered in O_p by balls $K(t,d_t)$ with centre t and radius d_t and with $d_t^{-n/2p} M_m(t) / M_p(\xi) \leq A < \infty$ for $t \in O_m$ and $\xi \in K(t,d_t) \subset O_p$.

This implies that the restriction map $T_{p,m}$ from $W^p_2(M_p;O_p)$ into $W^m_\infty(M_m;O_m)$ is continuous. Hence in virtue of F.6 W(O) is a projective limit of Hilbert spaces; it also follows (see [12]) that $S_{l,m} \circ T_{p,l}$ ($p>l>m$) is a Hilbert-Schmidt type map. Hence HS_1 and HS_2 ensure that W(O) is a nuclear $F\bar{S}$-space.

G.4. Let $\bar{O}$ be the closure of an open convex set in $\mathbb{R}^n$ and let a(m) and b(m) be two non-decreasing sequences, where for all m at least one of the inequalities $a_{m+1} > a_m$ and $b_{m+1} > b_m$ is valid. Suppose $M_m = M_{a(m),b(m)}$. Then for all $\varepsilon > 0$ and m=1,2,... there is an $R = R(\varepsilon,m) > 0$ such that, for all $\xi \in \bar{O}$ with $\|\xi\| \geq R$, $M_m(\xi) \leq \varepsilon M_{m+1}(\xi)$ (in other words $M_m(\xi)/M_{m+1}(\xi)$ approaches 0 as ξ tends to infinity in $\bar{O}$). According to D.3, $W(\bar{O}) = \text{proj lim}_m W^m_{\infty,0}(M_m;\bar{O})$ is an $F\bar{S}$-space and $W(\bar{O})$ also is the projective limit of $W^m_\infty(M_m;\bar{O})$.

G.5. According to F.12 the dual $W'(\bar{O})$ is the LS-space $\text{ind lim}_m (W^m_\infty(M_m;\bar{O}))'$. As in D.1 we have that a C^∞-function on a closed set F, whose derivatives are uniformly continuous and bounded on F, can be extended to a C^∞-function on $\mathbb{R}^n$ (see [10]), which is bounded on every ε-neighborhood of F. Hence the restriction map I from the Fréchet spaces $W(\mathbb{R}^n)$ into $W(\bar{O})$ is surjective. As in D.2 two things follow: firstly $W'(\bar{O})$ is a closed linear subspace of $W'(\mathbb{R}^n)$ and secondly $W'(\bar{O})$ is the space of all elements of $W'(\mathbb{R}^n)$ with support in $\bar{O}$. [4)]

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4) The conditions on $\bar{O}$, mentioned in the footnotes on pages 8 and 9, also imply the second statement, from which the first statement and the surjectivity of I follow.

G.6. It is clear that the functions M_m satisfy the conditions HS_1 and HS_2 in G.3 with O_m replaced by $\mathbb{R}^n$, so that $d_t = d$ can be taken independent of t. Therefore $W(\mathbb{R}^n)$ is nuclear. Hence its dual $W'(\mathbb{R}^n)$ and the linear subspace $W'(\bar{O})$ are nuclear. Since $W'(\bar{O})$ is reflexive, also $W(\bar{O})$ is nuclear.

G.7. The same properties as in G.1, G.2 and G.3 are valid for A spaces: let Ω_m be an increasing (decreasing) sequence of domains in $\mathbb{C}^n$ with union (intersection) Ω and let M_m be an increasing (decreasing) sequence of continuous positive weightfunctions on Ω_m. Condition HS_1 is:

$$(HS_1) \qquad \forall m, \exists p > m \; (\forall p, \exists m > p) \text{ such that } \int_{\Omega_m} \left(\frac{M_m(z)}{M_p(z)}\right)^2 dx\, dy \leq \infty.$$

Condition HS_2 is: $\forall m, \exists p > m$ $(\forall p, \exists m>p)$ such that

Ω_m can be covered in Ω_p by polydiscs $D(z,d_z) = \{w \mid |w_i - z_i| < d_z, i=1,\dots,n\}$ with $d_z^{-n} M_m(z)/M_p(w) \leq A < \infty$ for $z \in \Omega_m$ and $w \in D(z,d_z) \subset \Omega_p$.

Then HS_1 and HS_2 ensure that a projective (inductive) limit of $A_\infty(M_m;\Omega_m)$ spaces is a nuclear $F\bar{S}$-space (nuclear LS-space). Here we have the Hilbert spaces $A_2(M_m;\Omega_m)$ of all in Ω_m holomorphic functions f, for which $M_m f$ belongs to $L^2(\Omega_m)$ with inproduct

$$(f,g) = \int_{\Omega_m} M_m^2(z)\, f(z)\cdot\overline{g(z)}\, dx\, dy.$$

HS_1 implies that the embedding of $A_\infty(M_p;\Omega_p)$ into $A_2(M_m;\Omega_m)$ is continuous and HS_2 implies that the embedding of $A_2(M_p;\Omega_p)$ into $A_\infty(M_m;\Omega_m)$ is continuous.

G.8. Let Ω_k be an increasing sequence of domains in $\mathbb{C}_n$ with union Ω and let M_m be a decreasing sequence of continuous positive weightfunctions on Ω. Let us denote $\text{proj lim}_k\, A_2(M_m;\Omega_k)$ by $A_2(M_m;\Omega)$ and the closure of this space in $A_2(M_m;\Omega_k)$ by $A_2(M_m;\Omega)^{\bar{k}}$ and let us take the same notations for the A_∞-spaces. Assume that we have already proved that for some positive integer q all spaces $A_\infty(M_m;\Omega)^{\bar{k}}$ (m=1,2,.. and

$k=1,2,\ldots)$ can be continuously embedded into $A_\infty(M_{m+q};\Omega_l)$ for all $l \geq k$. Furthermore, suppose that the conditions HS_1 and HS_2 of G.7 are satisfied, so that for all k and some r and t > 0, there are continuous maps from $A_\infty(M_m;\Omega_k)$ into $A_2(M_{m+r};\Omega_k)$ and from this last space into $A_\infty(M_{m+r+t};\Omega_{k-1})$. In particular it follows that the identity map from $A_2(M_m;\Omega_m)$ into $A_2(M_{m+r+t};\Omega_{m-1})$ is a Hilbert-Schmidt type map. This remains true if we restrict this map to $A_2(M_m;\Omega)^{\overline{m}}$. Then the range is contained in $A_2(M_{m+r+t};\Omega)^{\overline{m-1}}$, which can continuously be embedded successively into $A_\infty(M_{m+r+2t};\Omega)^{\overline{m-2}}$, into $A_\infty(M_{m+r+2t+q};\Omega)^{\overline{m+2r+2t+q}}$ and into $A_2(M_{m+2r+2t+q};\Omega)^{\overline{m+2r+2t+q}}$. Hence the space $\tilde{H} = \text{ind lim}_m \underline{A}_\infty (M_m;\Omega)^{\overline{m}}$ is also the inductive limit of the Hilbert spaces $A_2(M_m;\Omega)^{\overline{m}}$, where the embedding maps are of Hilbert-Schmidt type. Thus $\tilde{H}$ is nuclear.

G.9. Projective and inductive limits of nuclear spaces are nuclear.

H. We give some special examples of function spaces and their duals, namely the usual spaces of distributions; they can be found in [8] or [12].

H.1. We define the LS-space S' of tempered distributions as the dual of the F$\bar{S}$-space $S = \text{proj lim}_m W^m_\infty((1+\|\xi\|)^m,\mathbb{R}^n) = \text{proj lim}_m W^m_{\infty,0}((1+\|\xi\|)^m;\mathbb{R}^n)$. We denote $W^m_{\infty,0}((1+\|\xi\|)^m,\mathbb{R}^n)$ with the norm $\|\cdot\|_m$ by S_m and $W^m_{\infty,0}((1+\|\xi\|)^p;\mathbb{R}^n)$ with the norm $\|\cdot\|^p_m$ by S^p_m. Sometimes we write S'_ξ and S_ξ in order to express that the distributions f, in that case also written as f_ξ, act on functions ϕ of the variable ξ; this action is written as $\langle f,\phi\rangle$, $\langle f,\phi\rangle_S$, $\langle f_\xi,\phi(\xi)\rangle$, $\langle f,\phi\rangle_\xi$ or $\langle f_\xi,\phi(\xi)\rangle_\xi$.

The Fourier transform F is an isomorphism of S onto S and we have

$$F[\phi](x) \overset{\text{def}}{=\!=} \int_{-\infty}^{\infty} e^{ix\cdot\xi}\, \phi(\xi)d\xi$$

and $\|F[\phi]\|^m_p \leq c_r \|\phi\|^{p+n+r}_m$ for any $r > 0$ and $c_r > 0$ depending on r. We define the Fourier transform of distributions f in S' by

$$\langle F[f], \phi\rangle = \langle f, F[\phi]\rangle$$

for all $\phi \in S$. We will always use the square brackets $F[\cdot]$ for the Fourier transform in S'.

H.2. Let O be open in $\mathbb{R}^n$ and K_m an increasing sequence of relatively compact open subsets of O with $\bar{K}_m \subset K_{m+1} \subset\subset O$ for $m=1,2,\ldots$ and $\cup_m K_m = O$. We define the LS-space $E'(O)$ of distributions with compact support in O as the dual of the F$\bar{\text{S}}$-space $E(O) = \text{proj lim}_m\, W_\infty^m(1;\bar{K}_m)$. If $O = \mathbb{R}^n$ we just write E' and E. We denote $W_\infty^m(1;\bar{K}_m)$ with the norm $\|\cdot\|_m$ or $\|\cdot\|_m^E$ by $E_m(\bar{K}_m)$ and $W_\infty^m(1;\bar{K}_k)$ with the norm $\|\cdot\|_{m,k}$ or $\|\cdot\|_{m,k}^E$ by $E_m(\bar{K}_k)$.

H.3. Let $O \subset \mathbb{R}^n$ and $\{K_k\}_{k=1}^\infty$ be as in H.2. We define the space $D'(O)$ of distributions in O as the dual of the strict inductive limit $D(O) = \text{ind lim}_k\, D(\bar{K}_k)$ with $D(\bar{K}_k)$ the F$\bar{\text{S}}$-space of C^∞-functions with compact support in $\bar{K}_k$ provided with the projective limit topology induced by the norms $\|\phi\|_{m,k} = \|\phi\|_{m,k}^D \overset{\text{def}}{=} \sup\{|D^\alpha\phi(\xi)| \mid \xi \in K_k, |\alpha| \le m\}$. We denote by $D_m(\bar{K}_k)$ the closure of $D(\bar{K}_k)$ in the norm $\|\cdot\|_{m,k}$ and we write D' and D, if $O = \mathbb{R}^n$.

H.4. The space Z consisting of entire functions is defined as $Z = \text{ind lim}_k\, \text{proj lim}_m\, A_\infty((1+\|z\|)^m e^{-k\|y\|};\mathbb{C}^n)$. The Fourier transform is an isomorphism between D and Z. The dual of Z is denoted by Z'.

3. ENTIRE FUNCTIONS

In this section we give appropriate topologies to the spaces occuring in the theorem of Paley-Wiener-Schwartz.

In the sequel K will be a compact convex set in $\mathbb{R}^n$ and O an open convex neighborhood of K in $\mathbb{R}^n$. We define the function

$$I_K(y) = \sup_{\xi\in K} -y\cdot\xi.$$

Since K is convex we have $\{\xi \mid -y\cdot\xi \le I_K(y),\ \forall y \in \mathbb{R}^n\} = K$ and if $0 \in K$, then $I_K(y) \geq 0$. If K is the ball with centre 0 and radius a in $\mathbb{R}^n$, then $I_K(y) = a\|y\|$.

The definition of the Fourier transform f of a distribution $g \in \mathcal{E}'$ or $\mathcal{E}'(O)$ (H.2) given in H.1 agrees with

$$f(x) = \langle g_\xi, e^{ix\cdot\xi}\rangle \qquad \text{(see [8, prop. 29.1]).}$$

According to the theorem of Paley-Wiener-Schwartz [8, theorem 29.2] the Fourier transform f of a distribution $g \in \mathcal{E}'(O)$ with support contained in K can be extended to the complex space $\mathbb{C}^n$ as an entire function satisfying

$$(3.1) \qquad |f(z)| \leq M(1+\|z\|)^m e^{I_K(y)} \qquad \text{for all } z = x + iy \in \mathbb{C}^n,$$

where M is a certain positive constant and m an integer, both depending on f. Conversely such a function is the Fourier transform of a distribution with support in K. This extended Fourier transform f of g is given by

$$(3.2) \qquad f(z) = \langle g_\xi, e^{iz\cdot\xi}\rangle \qquad \text{(see [8, prop. 29.1]).}$$

We provide the space of entire functions f that satisfy (3.1) with a topology such that the Fourier transform $\mathcal{F}$ is an isomorphism (a bijective map in both directions continuous). Let $\{K_m\}_{m=1}^{\infty}$ be an increasing sequence of open relatively compact convex sets in $\mathbb{R}^n$ with $\bar{K}_m \subset K_{m+1} \subset\subset O$ for m=1,2,... and $\bigcup_{m=1}^{\infty} K_m = O$, where O is an open convex set in $\mathbb{R}^n$ (not necessarily with compact closure). We define the Banach space $H^m(K_m)$ according to B.4 by

$$H^m(K_m) \stackrel{\text{def}}{=\!=} A_\infty((1+\|z\|)^{-m} e^{-I_{K_m}(y)}; \mathbb{C}^n)$$

with the norm $\|\cdot\|^m$. From C.7 it follows that the identity map from $H^m(K_m)$ into $H^{m+1}(K_{m+1})$ is compact, so that

$$\tilde{H}(O) \stackrel{\text{def}}{=\!=} \operatorname*{ind\,lim}_{m\to\infty} H^m(K_m)$$

is an LS-space (see F.8). Therefore $\tilde{H}(0)$ is a bornological (F.11), regular (F.16) and complete (F.14) Montel space (F.10). Moreover, according to G.7 $\tilde{H}(0)$ is nuclear.

The space $E'(0)$ of distributions with compact support in 0 has been given in H.2 and it follows from F.12 that $E'(0) = \text{ind lim}_m\ (E_m(\bar{K}_m))'$. Now we state and prove the main theorem of this section.

Theorem 3.1. *The map* F: $E'(0) \to H(0)$ *given by* $F(g)(z) = \langle g_\xi, e^{iz\cdot\xi}\rangle$ *for* $g \in E'(0)$ *is an isomorphism.*

Proof. The bijectivity of F is the theorem of Paley-Wiener-Schwartz. In order to prove the continuity of F it is sufficient to prove that F is a bounded map, since $E'(0)$ is bornological (see F.11). So let $B \subset E'(0)$ be a bounded set, that means according to F.15 and F.16 that $B \subset (E_m(\bar{K}_m))'$ and for all $g \in B$ $\|g\|_{-m} \leq M$ for some integer m and positive constant M. For the images $F(g) = f$ we have

$$|f(z)| = |\langle g_\xi, e^{iz\cdot\xi}\rangle| \leq \|g\|_{-m} \|e^{iz\cdot\xi}\|_m^E \leq M(1+\|z\|)^m e^{I_{K_m}(y)}$$

for all $y \in \mathbb{C}^n$. Thus

$$\|f\|^m = \sup_{z\in\mathbb{C}^n} (1+\|z\|)^{-m} e^{-I_{K_m}(y)} |f(z)| \leq M ,$$

so that $f \in H^m(K_m)$ and $F(B)$ is bounded in $H^m(K_m)$, thus bounded in $\tilde{H}(0)$.

In order to prove the continuity of F^{-1} it is again sufficient to prove that, for each m, F^{-1} is a bounded map from $H^m(K_m)$ into $(E_{m+n+1}(\bar{K}_{m+n+1}))'$, since $\tilde{H}(0)$ is bornological and regular. Let $A \subset H^m(K_m)$ be a bounded set; thus there exists a positive constant M such that for all $f \in A$

$$(3.3) \qquad |f(z)| \leq M(1+\|z\|)^m e^{I_{K_m}(y)} \qquad \text{for all } z \in \mathbb{C}^n.$$

For all $\phi \in S$ (see H.1) we get for the images $g = F^{-1}(f) \in E'(0)$

$$|\langle g,\phi\rangle| = |\langle f, F^{-1}[\phi]\rangle| \leq M \int_{-\infty}^{\infty} (1+\|x\|)^m |F^{-1}[\phi](x)|\, dx \leq$$

$$\leq MC\,||\,\mathcal{F}^{-1}[\phi]\,||_0^{m+n+1} \leq MCc_1\,||\,\phi\,||_{m+n+1}^{n+1}\,.$$

Thus $\mathcal{F}^{-1}$ is a bounded map from $H^m(K_m)$ into $(S_{m+n+1}^{n+1})'$. The range of this map consists of distributions in S' with support in K_m according to the theorem of Paley-Wiener-Schwartz, so the range of $\mathcal{F}^{-1}$ is contained in $(\mathcal{E}_{m+n+1}(\bar{K}_m))'$, because K_m is convex (see the remark in D.2). From D.2 it follows that $(\mathcal{E}_{m+n+1}(\bar{K}_m))'$ is a linear subspace of $(S_{m+n+1}^{n+1})'$, thus $\mathcal{F}^{-1}$ is a bounded map from $H^m(K_m)$ into $(\mathcal{E}_{m+n+1}(\bar{K}_m))'$ and certainly into $(\mathcal{E}_{m+n+1}(\bar{K}_{m+n+1}))'$. □

Remark 3.1. In fact we have shown that
(3.4) $\mathcal{F}^{-1}$ is a continuous map from $H^m(K_m)$ into $(\mathcal{E}_{m+n+1}(\bar{K}_m))'$, and that
(3.5) $\mathcal{F}$ is a continuous map from $(\mathcal{E}_{m+n+1}(\bar{K}_m))'$ into $H^{m+n+1}(K_m)$.

Remark 3.2. In vieuw of F.6 and remark 3.1 we see that $\mathcal{F}$ is an isomorphism also between the LS-spaces $(\mathcal{E}(\bar{K}))' \stackrel{\text{def}}{=} \text{ind lim}_m\ (\mathcal{E}_m(\bar{K}))'$ and $H(K) \stackrel{\text{def}}{=} \text{ind lim}_m\ H^m(K)$, where K is an open relatively compact set in $\mathbb{R}^n$; that $(\mathcal{E}(\bar{K}))'$ is an LS-space follows from D.3 too. In G.5 is stated that $(\mathcal{E}(\bar{K}))'$ is a closed linear subspace of S'. Therefore $(\mathcal{E}(\bar{K}_k))'$ is a closed linear subspace of $(\mathcal{E}(\bar{K}_{k+1}))'$ and the inductive limit $E' \stackrel{\text{def}}{=} \text{ind lim}_k\ (\mathcal{E}(\bar{K}_k))'$ is strict. It follows from F.4 and F.5 that a continuous map from E' into $\mathcal{E}'(0)$ exists and from F.5 that a continuous map from $\mathcal{E}'(0)$ into E' exists. Thus according to F.6 $E' = \mathcal{E}'(0)$ and since for the same reason $\mathcal{F}$ is an isomorphism between E' and $\tilde{H} \stackrel{\text{def}}{=} \text{ind lim}_k\ H(K_k)$,

(3.6) $$\tilde{H}(0) = \underset{k\to\infty}{\text{ind lim}}\ \underset{m\to\infty}{\text{ind lim}}\ H^m(K_k)\ ,$$

where the inductive limit for $m \to \infty$ yields an LS-space and the inductive limit for $k \to \infty$ is a strict inductive limit of complete spaces.

In part II, where entire functions of exponential growth in $||z||$ will be treated, we will see that it is not possible to separate the original inductive limit into two inductive limits. This difference is closely related to the difference between the concepts support of a distribution and carrier of an analytic functional.

4. NEWTON SERIES FOR ENTIRE FUNCTIONS

In this section we derive the Newton series (1.1) for entire functions satisfying (3.3).

For the series (1.3) to be convergent it is sufficient that for $h \in \mathbb{C}^n$ $|e^{-h\cdot\zeta}-1| < 1$. If we denote the real and imaginary part of $-h\cdot\zeta$ by u and v: $-h\cdot\zeta = u + iv$, we get

$$(e^{u+iv}-1)(e^{u-iv}-1) = e^{2u} - 2e^{u}\cos v + 1 < 1 \quad ,$$

so for $u \neq -\infty$: $e^u < 2\cos v$ or (see fig. 4.1)

(4.1) $$u < \log(2\cos v)$$

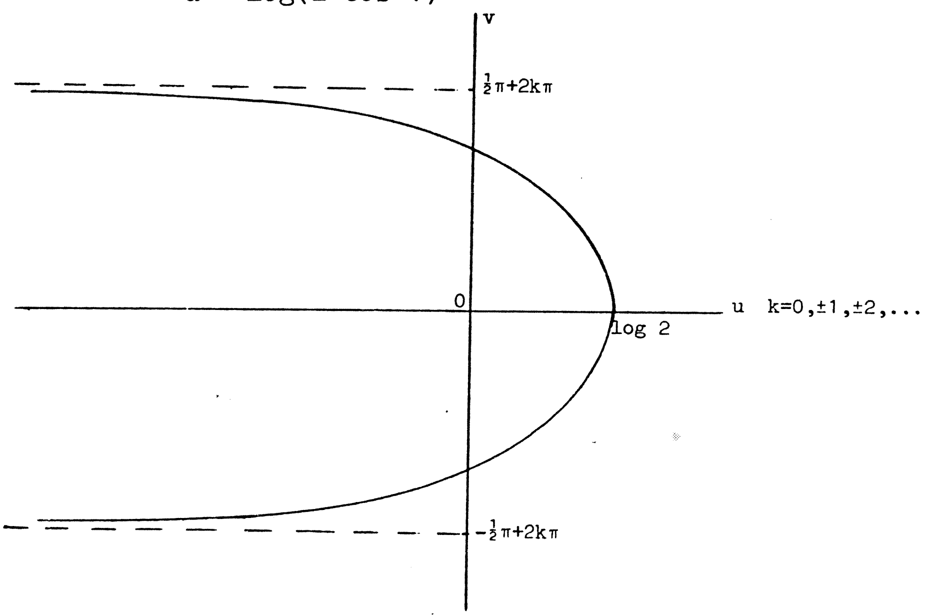

Figure 4.1

For the carrier K of the Fourier transform of f we have

(4.2) $$K \subset \Omega_h = \{\zeta \mid |e^{-h\cdot\zeta}-1| < 1\} \subset \mathbb{C}^n$$

and for the support K of the Fourier transform g of f, in case g is a distribution, we have

(4.3) $$K \subset O_h = \{\xi \mid |e^{-h\cdot\xi}-1| < 1\} \subset \mathbb{R}^n.$$

For any $\xi \in O_h$ and $s \in \mathbb{C}$

$$\lim_{N\to\infty} \sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\xi}-1)^k = e^{-sh\cdot\xi} \in E(O_h)$$

and, for all m, $|e^{-h\cdot\xi}-1| \le \rho_m < 1$ when $\xi \in K_m \subset O_h$. Thus

$$\|\sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\xi}-1)^k\|_m^E \le c_1(m) \sum_{k=m}^{N} \binom{|s|+k-1}{k} k(k-1)\dots(k-m+1)\rho_m^k \le c_2(m)$$

independent of N, because

$$|\binom{s}{k}| = |s(s-1)\dots(s-k+1)/k!| \le |s|(|s|+1)\dots(|s|+k-1)/k! =$$

$$= \binom{|s|+k-1}{k}$$

and the series converges according to d'Alemberts criterium:

$$\lim_{k\to\infty} \rho_m \frac{\binom{|s|+k}{k+1}(k+1)}{\binom{|s|+k-1}{k}(k-m+1)} = \rho_m \lim_{k\to\infty} \frac{|s|+k}{k-m+1} = \rho_m < 1.$$

Using G.2 we come to

Lemma 4.1. *The sequence* $\left\{\sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\xi}-1)^k\right\}_{N=1}^{\infty}$ *converges in* $E(O_h)$.

Let f be an element of $H(O_h)$ and let $g \in E'(O_h)$ be its inverse Fourier transform. With the aid of theorem 3.1 and lemma 4.1 we derive the Newton series for $z,h \in \mathbb{C}^n$ and $s \in \mathbb{C}$:

(4.4)
$$f(z+ish) = \langle g_\xi, e^{iz\cdot\xi-sh\cdot\xi}\rangle = \langle g_\xi, \lim_{N\to\infty} e^{iz\cdot\xi} \sum_{k=0}^{N} \binom{s}{k}(e^{-h\cdot\xi}-1)^k\rangle =$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} \binom{s}{k} \langle g_\xi, e^{iz\cdot\xi}(e^{-h\cdot\xi}-1)^k\rangle =$$

$$= \sum_{k=0}^{\infty} \binom{s}{k} \langle g_\xi, \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} e^{i(z+imh)\cdot\xi}\rangle =$$

$$= \sum_{k=0}^{\infty} \binom{s}{k} \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} f(z+imh) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta_{ih}^k f(z)$$

according to (1.2). Considering

$$(4.5) \qquad \phi_N(\xi) \stackrel{\text{def}}{=\!=} \sum_{k=0}^{N} \binom{s}{k} (e^{-h\cdot\xi}-1)^k$$

as a multiplicator in $E'(O_h)$ we see from lemma 4.1 that for any $g \in E'(O_h)$ the sequence $\{g_\xi \phi_N(\xi)\}_{N=1}^{\infty}$ converges weakly in $E'(O_h)$ and since $E(O_h)$ is a Montel space, it converges strongly too (F.10). Therefore according to theorem 3.1 the Newton series (4.4) converges in the topology of $\tilde{H}(O_h)$. It follows that (4.4) certainly converges uniformly in z on compact subsets of $\mathbb{C}^n$, which is the convergence given in [3].

If in addition one wants to know of all the norms that determine the topology of $H(O_h)$ the finest one (the largest), in which the series (4.4) converges, the convergence of the sequence $g_\xi \phi_N(\xi)$ needs to be analysed more precisely. If $g \in (E_m(\bar{K}_k))'$ the limit of this sequence also belongs to $(E_m(\bar{K}_k))'$. Denoting the limit $e^{-sh\cdot\xi}$ of $\phi_N(\xi)$ by $\phi(\xi)$ and remarking that for each m there is a constant C_m such that for any two functions ψ and $\chi \in E(O_h)$

$$\|\psi\cdot\chi\|^{E}_{m,k} \leq C_m \|\psi\|^{E}_{m,k} \cdot \|\chi\|^{E}_{m,k} ,$$

we get for every $\psi \in E(O_h)$ with the aid of lemma 4.1

$$|\langle g\phi_N - g\phi, \psi\rangle| = |\langle g, (\phi_N-\phi)\psi\rangle| \leq$$

$$\leq \|g\|_{-(m,k)} C_m \|\phi_N-\phi\|^{E}_{m,k} \|\psi\|^{E}_{m,k} \leq \varepsilon \|\psi\|^{E}_{m,k}$$

for N sufficiently large. Hence the sequence $\{g_\xi \phi_N(\xi)\}_{N=1}^{\infty}$ converges strongly in $(E_m(\bar{K}_k))'$. From this and from (3.4) and (3.5) we derive that the series (4.4) with f satisfying (3.3) converges according to:
$\forall \varepsilon > 0$, $\exists N_0(\varepsilon)$, $\forall z \in \mathbb{C}^n$ and $\forall N \geq N_0$

$$(4.6) \qquad \left|f(z+ish) - \sum_{k=0}^{N} \binom{s}{k} \Delta^k_{ih} f(z)\right| < \varepsilon(1+\|z\|)^{m+n+1} e^{I_{K_m}(y)} .$$

The convergence is absolute too, for

$$\left|\binom{s}{k}\, \Delta^k_{ih} f(z)\right| = \left|\binom{s}{k}\right| \; |\langle g_\xi, e^{iz\cdot\xi}(e^{-h\cdot\xi}-1)^k\rangle| \le$$
$$\le c\, \|g\|_{-r}(1+\|z\|)^r e^{I_{K_r}(y)} \left|\binom{s}{k}\right| k(k-1)\ldots(k-r+1)\rho_r^k$$

with $r = m+n+1$ and the right-hand side of this inequality is the general term of an absolutely converging sequence.

Thus we have proved

Theorem 4.1. *For* f *satisfying (3.3) with* $K_m \subset O_h$, *where* O_h *is given by (4.3), the Newton series (4.4) is valid; the series converges absolutely in the topology of* $H(O_h)$ *or more precisely according to (4.6).*

We can interprete the results we have obtained in a different way. $I_K(y)$ is a convex homogeneous function of $y \in \mathbb{R}^n$ determined by the compact convex set $K \subset \mathbb{R}^n$; conversely an arbitrary convex continuous function a on the unit sphere pr $\mathbb{R}^n$ (this means that the homogeneous function $\tilde{a}(y) = \|y\|\, a(\tilde{y})$ of $y \in \mathbb{R}^n$ is convex, see A.5 and for the notations A.1 and A.3) determines a compact convex set K by

$$(4.7) \qquad K = \{\xi \mid -y\cdot\xi \le a(y) \quad \text{for all } y \in \mathbb{R}^n \text{ with } \|y\| = 1\}.$$

Now let the entire function f be given such that for all $z \in \mathbb{C}^n$

$$(4.8) \qquad |f(z)| \le M(1+\|z\|)^m e^{\tilde{a}(y)}\;;$$

f is the Fourier transform of a distribution with support in K, where K is given by (4.7). From (4.1) and (4.3) it follows that for f satisfying (4.8) the Newton series (4.4) is valid, if $h = h_1+ih_2 \in \mathbb{C}^n$ is such that

$$\tilde{a}(h_1) < \log(2\cos\tilde{a}(\pm h_2)) \text{ and } -\tfrac{1}{2}\pi + 2k\pi < \tilde{a}(\pm h_2) < \tfrac{1}{2}\pi + 2k\pi$$

for some integer k, where the inequalities must hold for both the + and the

- sign, because $-\tilde{a}(-h) \leq -h\cdot\xi \leq \tilde{a}(h)$ for $\xi \in K$. In case $h \in \mathbb{R}^n$ is real, this means that the Newton series is valid for all vectors h such that for any $y \in \mathrm{pr}\, C$, whenever $h = \|h\| y$,

$$\|h\| < \frac{\log 2}{a(y)} \qquad \text{if } a(y) > o \qquad \text{or}$$

$$\|h\| \text{ arbitrarily large} \qquad \text{if } a(y) \leq 0 .$$

5. POSSIBLE GENERALIZATIONS

I. If ζ varies in a compact set of Ω_h (see (4.2)), the series (1.3) and its derivatives converge uniformly. We remark that also for ζ in non-compact sets of Ω_h the series (1.3) and its derivatives converge. We investigate how this convergence depends on ζ.

First let us estimate the Newton polynomial $\binom{s}{k}$ in $s \in \mathbb{C}$:

$$\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{(k-1)!}\,\frac{1}{k} = \frac{(-1)^{k-1}}{k}\, s(1-s)(1-\frac{s}{2})\dots(1-\frac{s}{k-1}) =$$

$$= \frac{(-1)^{k-1}}{k}\, s \left\{\prod_{r=1}^{k-1}(1-\frac{s}{r})e^{\frac{s}{r}}\right\} e^{-s(1+\frac{1}{2}+\dots+\frac{1}{k-1})} .$$

For $k \to \infty$ the factor between brackets becomes Euler's infinite product, which equals

$$\frac{1}{\Gamma(-s)}\,\frac{1}{-s}\, e^{\gamma^{s}} ,$$

where γ is Euler's constant

$$\gamma = \lim_{k\to\infty} (1+\tfrac{1}{2}+\dots+\frac{1}{k-1} - \log k) \qquad \text{(see [5]).}$$

Here $1/\Gamma(-s)$ is an entire function. Furthermore

$$\frac{(-1)^{k-1}}{k}\, s \left[\frac{1}{\Gamma(-s)}\,\frac{1}{-s}\, e^{\gamma s}\right] e^{-s(\gamma+\log k)} = \frac{(-1)^k}{k^{s+1}}\,\frac{1}{\Gamma(-s)} ,$$

so that $|\Gamma(-s)|k^{\mathrm{Re}\ s+1}|\binom{s}{k}| \to 1$ as $k \to \infty$.
Thus there is an index $N_1(s)$ depending on s such that for $\mathrm{Re}\ s \geq -\alpha$ with $\alpha \geq 0$

$$(5.1) \qquad |\binom{s}{k}| \leq \frac{2}{|\Gamma(-s)|} k^{\alpha-1} \qquad \text{for all } k \geq N_1(s)$$

Moreover, there is a constant B_s depending on s such that for $\mathrm{Re}\ s \geq -\alpha$

$$(5.2) \qquad |\binom{s}{k}| \leq B_s\, k^{\alpha-1} \quad \text{for all } k=1,2,\ldots \quad \text{and } \binom{s}{k} = 1 \text{ for } k = 0.$$

For s in a compact set in $\mathbb{C}$, $N_1(s)$ and $1/|\Gamma(-s)|$ are uniformly bounded.

According to (4.1) and (4.2) we can write Ω_h as

$$\Omega_h = \{\zeta \mid -\mathrm{Re}\ h\cdot\zeta < \log(2\cos \mathrm{Im}\ h\cdot\zeta)\} \subset \mathbb{C}^n$$

and for $0 < \varepsilon \leq \frac{1}{2}$ we define

$$\Omega(\varepsilon) = \{\zeta] - \mathrm{Re}\ h\cdot\zeta < \log(2\cos \mathrm{Im}\ h\cdot\zeta - \varepsilon)\} \subset \Omega_h.$$

where ε may depend on ζ provided that $0 < \varepsilon(\zeta) \leq \frac{1}{2}$ for all $\zeta \in \Omega_h$.
For $\zeta \in \Omega(\varepsilon)$ we have

$$]1 - e^{-h\cdot\zeta}] = \sqrt{1 + e^{2(-\mathrm{Re}\ h\cdot\zeta)} - 2e^{-\mathrm{Re}\ h\cdot\zeta}\cos \mathrm{Im}\ h\cdot\zeta} \leq$$

$$\leq \sqrt{1 + e^{2(-\mathrm{Re}\ h\cdot\zeta)} - e^{-\mathrm{Re}\ h\cdot\zeta}(e^{-\mathrm{Re}\ h\cdot\zeta} + \varepsilon)} \leq$$

$$\leq \sqrt{1 - \varepsilon e^{-\mathrm{Re}\ h\cdot\zeta}} \leq 1 - \tfrac{1}{2}\varepsilon e^{-\mathrm{Re}\ h\cdot\zeta} \overset{\mathrm{def}}{=\!=} \rho\ ,$$

so that $\frac{1}{2} \leq 1 - \varepsilon < \rho < 1$. Hence the function $\phi_N(\xi)$ defined in (4.5) can be extended to a function $\phi_N(\zeta)$ holomorphic in Ω_h and we get for $\mathrm{Re}\ s \geq -\alpha$ according to (5.2)

$$|\phi_N(\zeta)| \le 1 + B_s \sum_{k=1}^{N} \rho^k k^{\alpha-1} \le$$

$$\le \begin{cases} 1 + B_s \int_0^\infty e^{t\log \rho} t^{\alpha-1} dt = 1 + B_s\Gamma(\alpha)/(-\log \rho)^\alpha, & \text{in case } \alpha > 0 \\ 1 + B_s\{-\log(1-\rho)\} & \text{, in case } \alpha = 0. \end{cases}$$

For $\zeta \in \Omega(\varepsilon)$ we get, respectively,

$$1 + \frac{B_s\Gamma(\alpha)}{(-\log \rho)^\alpha} \le 1 + \frac{B_s\Gamma(\alpha)}{\{-\log(1-\frac{1}{2}\varepsilon e^{-\mathrm{Re}\ h\cdot\zeta})\}^\alpha} \le 1 + B_s\Gamma(\alpha)(\tfrac{2}{\varepsilon})^\alpha e^{\alpha \mathrm{Re}\ h\cdot\zeta} \le$$

$$\le C(s,\alpha) \frac{1}{\varepsilon^\alpha} e^{\alpha \mathrm{Re}\ h\cdot\zeta} \qquad \text{for } \alpha > 0,$$

$$1 + B_s\{-\log(1-\rho)\} \le 1 - B_s \log \tfrac{1}{2}\varepsilon + B_s ||h|| \cdot ||\zeta|| \le$$

$$\le C(s) \log \frac{1}{\varepsilon} \cdot (1+||\zeta||) \qquad \text{for } \alpha = 0 .$$

In order to estimate the derivatives we note that $\phi_N(\zeta)$ is holomorphic in Ω_h and that there is an $\varepsilon_1 > 0$ such that

$$\{\zeta \mid |\zeta-\zeta_0| < \varepsilon_1, \zeta_0 \in \Omega(\varepsilon)\} \subset \Omega(\tfrac{1}{2}\varepsilon) ,$$

namely for $\varepsilon_1 = \varepsilon/(6||h||)$ we have for $\zeta \in \mathbb{C}^n$ with $|\zeta-\zeta_0| < \varepsilon_1$ and $\zeta_0 \in \Omega(\varepsilon)$:– $\mathrm{Re}\ h\cdot\zeta \le -\mathrm{Re}\ h\cdot\zeta_0 + \varepsilon_1 ||h|| \le \log(2\cos \mathrm{Im}\ h\cdot\zeta_0 - \varepsilon) + \varepsilon_1 ||h|| \le$
$\le \log(2\cos \mathrm{Im}\ h\cdot\zeta + \varepsilon_1 ||h|| - \varepsilon) + \varepsilon_1 ||h|| = \log(2\cos \mathrm{Im}\ h\cdot\zeta - \frac{5}{6}\varepsilon) + \frac{1}{6}\varepsilon \le$

$$\le \log(2\cos \mathrm{Im}\ h\cdot\zeta) + \log(1 - \frac{5}{6}\frac{\varepsilon}{2\cos \mathrm{Im}\ h\cdot\zeta}) + \frac{2}{6}\frac{\varepsilon}{2\cos \mathrm{Im}\ h\cdot\zeta} \le$$

$$\le \log(2\cos \mathrm{Im}\ h\cdot\zeta) + \log(1 - \frac{3}{6}\frac{\varepsilon}{2\cos \mathrm{Im}\ h\cdot\zeta}) = \log(2\cos \mathrm{Im}\ h\cdot\zeta - \tfrac{1}{2}\varepsilon).$$

For the derivatives we obtain by means of Cauchy's formula the bounds

$$\sup_{\zeta\in\Omega(\varepsilon)} | D^k\phi_N(\zeta) | \le \sup_{\zeta\in\Omega(\varepsilon)} \left| \frac{k!}{(2\pi i)^n} \int\!\!\cdots\!\!\int_{|\zeta_i - z_i| = \frac{\varepsilon 1}{\sqrt{n}}} \frac{\phi_N(z)}{(\zeta-z)^{k+\bar{1}}} dz_1\ldots dz_n \right| \le$$

$$\le k! \left(\frac{\sqrt{n}}{\varepsilon_1}\right)^{|k|} \sup_{\zeta\in\Omega(\frac{1}{2}\varepsilon)} |\phi_N(\zeta)|,$$

where $k+\bar{1} = (k_1+1,\ldots,k_n+1)$.

Finally we have found constants C_i such that for Re $s \ge -\alpha$ and for every multi-index k

$$\forall\zeta \in \Omega(\varepsilon): \quad | D^k\phi_N(\zeta) | \le \begin{cases} C_1(s,k,\alpha) \dfrac{1}{\varepsilon^{\alpha+|k|}} e^{\alpha \mathrm{Re}\, h\cdot\zeta}, & \alpha > 0 \\ C_2(s,k) \dfrac{\log 1/\varepsilon}{\varepsilon^{|k|}} (1+||\zeta||), & \alpha = 0. \end{cases}$$

Now let us take $\varepsilon(\zeta) = \varepsilon_0/||\zeta||^\beta$ if $||\zeta|| > 1$ and $\varepsilon(\zeta) = \varepsilon_0$ if $||\zeta|| \le 1$ for some fixed ε_0, $0 < \varepsilon_0 \le \frac{1}{2}$ and $\beta \ge 0$, then we have

$$(5.3) \quad \forall\zeta \in \Omega(\varepsilon(\zeta)): |D^k\phi_N(\zeta)| \le \begin{cases} C_1'(s,k,\alpha)(1+||\zeta||)^{\alpha\beta+\beta|k|} e^{\alpha \mathrm{Re}\, h\cdot\zeta}, & \alpha > 0 \\ C_2'(s,k)(1+||\zeta||)^{1+\beta|k|}(1+\beta\log(1+||\zeta||)), & \alpha = 0. \end{cases}$$

For $N_0 \ge N_1(s)$ and all $N > N_0$ we have according to (5.1)

$$(5.4) \quad \forall\zeta \in \Omega(\varepsilon(\zeta)): \quad | D^k(\phi_N(\zeta) - \phi_{N_0}(\zeta))| \le$$

$$\le \begin{cases} C_1''(k,\alpha)\; A(s)(1+||\zeta||)^{\alpha\beta+\beta|k|} e^{\alpha \mathrm{Re}\, h\cdot\zeta}, & \alpha > 0 \\ C_2''(k)\; A(s)(1+||\zeta||)^{1+\beta|k|}(1+\beta\log(1+||\zeta||)), & \alpha = 0 \end{cases}$$

with $A(s) = \left|\dfrac{1}{\Gamma(-s)}\right|$.

II. Let us now investigate how we can generalize the results of sections 3 and 4.

a) First we take h a real vector in $\mathbb{R}^n$.

We consider distributions g in S' or $\mathcal{D}'$ for the Fourier transform of which we want to derive the Newton series (1.1). In any case the support of g has to be contained in the halfspace $\mathcal{O}_h = \{ \xi \mid - h\cdot\xi < a\}$ with a = log 2. As we want the Newton series (1.1) to be valid for several vectors h, say for $h \in V \subset \mathbb{R}^n$, the support of g must lie in $O_V = \{\xi \mid \forall h \in V: - h\cdot\xi < a\} = O_{O(V)}$, where $O(V)$ is the convex envelope of V, since $\sup\{- h\cdot\xi \mid h \in V\} = \sup\{- h\cdot\xi \mid h \in O(V)\}$. So if the Newton series is valid for $h \in V$, it is also valid for $h \in O(V)$. Therefore we take V an open convex set in $\mathbb{R}^n$. If h varies in V, $\tilde{h}$ varies in a set U on the unit sphere. U determines a convex cone C by

$$C = \{\xi \mid \forall y \in U, \lambda > 0: \xi = \lambda y\} ;$$

we have U = pr C (see A.3). Let $r(y) = \sup\{\lambda \mid \lambda y \in V\}$ for $y \in$ pr C, the largest length of a vector in V in the direction of y and let $a(y) = \log 2/r(y)$. Then we have

$$O_V = \{\xi \mid \forall y \in \text{pr}\, C: - y\cdot\xi < a(y)\} \stackrel{\text{def}}{=\!=} O(a;C)$$

in the notation that will be used in the sequel.

Suppose that there is a vector $v \in V$ with also $- v \in V$, then this is valid too for vectors with direction in a neighborhood W in pr C of $y_0 = \tilde{v} \in$ pr C. All $\xi \in O_V$ must satisfy

$$\forall y \in W: |y\cdot\xi| < \sup_{y \in W \cup -W} a(y),$$

so that O_V is bounded. That case has been treated in sections 3 and 4 and the convex envelope of U on the unit sphere is the unit sphere itself; therefore $C = \mathbb{R}^n \setminus \{0\}$.

In order to generalize section 4 we have to take V contained in a convex cone C, where $C \cup \{0\}$ does not contain a straight line through the origin. If h varies in C, while its length $||h||$ does not exceed a given number b, then $a(y) = \log 2/b$ is constant for $y \in \text{pr } C$, or if we keep $||h||$ constant, then $a(y) = \log 2/||h||$.

Corollary 5.1. *Let* C *be an open convex cone in* $\mathbb{R}^n$ *such that* $C \cup \{0\}$ *does not contain a straight line through the origin; we will derive the Newton series (1.1) with* $h \in C$ *for functions that are the Fourier transform of distributions in* S' or $\mathcal{D}'$ *with supports contained in*

$$O(\frac{\log 2}{||h||};C) = \{\xi \mid \sup_{y \in \text{pr } C} - y\cdot\xi < \frac{\log 2}{||h||}\}.$$

The Fourier transform f of a distribution g_ξ in $\mathcal{D}'$ or S' is a function analytic in $\mathbb{R}^n + iy_0$ if and only if $g_\xi e^{-y_0\cdot\xi} \in S'$ (see [11] 26.2). In case $g \in S'$ it is sufficient for this that the support K of g satisfies $\forall\xi \subset K$: $- y_0\cdot\xi \le a||y_0||$ for a certain number a, thus that $K \subset \{\xi \mid - y_0\cdot\xi \le a||y_0||\}$. Let f be holomorphic in $\mathbb{R}^n + iV$ with V an open neighborhood of y_0 in $\mathbb{R}^n$. As before we can take V convex (see also E.4) and let C be the cone determined by V. We must have

$$K \subset O_V = \{\xi \mid \forall y \in V: - y\cdot\xi \le a(y)||y||\}=\{\xi| \ \forall y \in \text{pr } C: - y\cdot\xi \le$$
$$\le a(y)\} =$$
$$= \{\xi \mid \forall y \in C: - y\cdot\xi \le a(\tilde{y})||y|| \stackrel{\text{def}}{=\!=} \tilde{a}(y)\} \stackrel{\text{def}}{=\!=} O(a;C),$$

where now a is a certain continuous function on V, thus on pr C too, such that $\tilde{a}$ is a convex function on C. For all $y \in C$ we have

$$\tilde{a}(y) = \sup_{\xi\in O_V} - y\cdot\xi = I_{O_V}(y).$$

We see that f is holomorphic in the tubular radial domain $T^C = \mathbb{R}^n + iC$.

Again if V contains a vector y_0 with $- y_0$ in V too, O_V is bounded and the convex envelope of U determines the cone $\mathbb{R}^n \setminus \{0\}$. Then f must be holomorphic in $\mathbb{R}^n + i(\mathbb{R}^n\setminus\{0\})$, from which it follows in virtue of Bochner's theorem (E.4), that f is an entire function. In that case we have

$$O_V = \{\xi \mid \forall y \in \mathbb{R}^n: - y\cdot\xi \le I_{O_V}(y)\},$$

so that the conditions of the theorem of Paley-Wiener-Schwartz appear (see section 3).

Corollary 5.2. *Let* C *be an open convex cone in* $\mathbb{R}^n$ *such that* $C \cup \{0\}$ *does not contain* a *straight line through the origin and let* a *be a continuous convex function on* pr C (see A.5). *We will consider Fourier transforms of distributions in* S', *or of distributions* g *in* $\mathcal{D}'$ *with* $g_\xi e^{-y\cdot\xi} \in S'$ *for each* $y \in C$, *with support contained in*

$$O(a;C) = \{\xi \mid \forall y \in \text{pr } C: - y\cdot\xi < a(y)\}. \tag{5.5}$$

Such Fourier transforms are functions holomorphic in T^C.

b) Now we consider vectors h in C^n; we set $h = h_1 + ih_2$ with $h_1,h_2 \in \mathbb{R}^n$. We still require that ξ belongs to a non-compact set K contained in $\Omega_h \cap \mathbb{R}^n = O_h$ (see (4.2) and (4.3)), that is $- h_1\cdot\xi < \log(2 \cos h_2\cdot\xi)$ according to (4.1). Therefore we cannot have that h_1 varies in all of $\mathbb{R}^n$, but again we take $h_1 \in C_1$ with C_1 an open convex cone, so that

$$K \subset O\left(\frac{\log 2}{||h_1||}\, ;\, C_1\right) \overset{\text{def}}{=\!=} O_1$$

as in corollary 5.1. Let h_2 vary in an open convex cone C_2, then K should be contained in the union of the sets $V_k = \{\xi \mid 2k\pi - \frac{1}{2}\pi < h_2\cdot\xi < 2k\pi + \frac{1}{2}\pi\}$ for $k=0,\pm1,\pm2,\ldots$ (compare figure 4.1).

If $C_2 = \mathbb{R}^n \setminus \{0\}$, V_k is relatively compact for each k and K is contained in a countable union of compact sets all contained in O_1, so that f must be some countable sum of entire functions, which is itself holomorphic in $\mathbb{R}^n + iC_1$.

If $\bar{C}_2 \subset C_1 \cup \{0\}$, $V_k \cap O_1$ is relatively compact and again K is contained in a countable union of compact sets all contained in O_1.

If C_2 is not really contained in C_1, K must lie in a countable union of non-compact sets each contained in $O(a_k;O(C_1 \cup C_2))$ for some functions a_k on pr $O(C_1 \cup C_2)$. Then f must be some countable sum of functions holomorphic in $\mathbb{R}^n + iO(C_1 \cup C_2)$, which is itself holomorphic in $\mathbb{R}^n + iC_1$.

We will not go into the details of these cases, since they follow from the theorems of the subsequent sections.

6. FUNCTIONS HOLOMORPHIC IN TUBULAR RADIAL DOMAINS HAVING BOUNDARY VALUES

We treat the generalization of the theorem of Paley-Wiener-Schwartz mentioned in corollary 5.2 with $g \in S'$ and give the occurring spaces topologies, such that we can generalize theorem 3.1.

Let C be an open convex cone in $\mathbb{R}^n$, where $C \cup \{0\}$ does not contain a straight line through the origin and let a be a continuous function on pr C such that $\tilde{a}$ is a convex function on C, see A.5 (a need not to be bounded on pr C). The next lemma can be found in [11, 26.4 theorem 2], where a is a positive constant, but the proof given there also holds, when a is a continuous convex function of y not necessarily positive.

Lemma 6.1. *The following statements are equivalent:*

(1) f, *holomorphic in* T^C, *has the property that positive numbers* α *and* β *exist, such that for all compact subcones* C' *of* C (see A.3) *there is a positive constant* M(C') *with* $|f(z)| \leq M(C')\ (1+||z||)^{\beta}(1+||y||^{-\alpha})\ e^{\tilde{a}(y)}$ *for all* $z = x + iy \in T^{C'}$.

(2) $f(z) = \mathcal{F}[g_\xi e^{-y\cdot\xi}](x)$, *where* g_ξ *is a distribution in* S' *with support* K *contained in* $\bar{O}(a;C) \subset \mathbb{R}^n$; *here* O(a;C) *is given by formula (5.5)*

Such functions f determine a class of functions in [11] denoted by $H_1(a;C)$. Another property of these holomorphic functions is stated in the following lemma:

Lemma 6.2. *Functions of the class* $H_1(a;C)$ *have boundary values on the distinguished boundary of* T^C *in* S'. *More precisely, for all* $\phi \in S$

$$\lim_{y\to 0, y\in C'} \int_{-\infty}^{\infty} f(x+iy)\phi(x)dx = \langle f^*,\phi\rangle_S \qquad \textit{for a } f^* \in S'$$

and this limit is independent of the path $y \to 0$ *provided that this is contained in an arbitrary compact subcone* $C' \subset\subset C$. *If the path* $y \to 0$ *is contained only in* C, f^* *is attained in* Z' (see H.4), *that is*

$$\lim_{y\to 0, y\in C} \int_{-\infty}^{\infty} f(x+iy)\psi(x)dx = \langle f^*,\psi\rangle_Z \qquad \textit{for all } \psi \in Z.$$

Proof. We briefly discuss the proof, that is given in more detail in [11, 26.3]. It is easily seen that $f(x+iy) \in Z'$ for all $y \in C$, when we define for $\psi \in Z$,

$$\langle f,\psi\rangle_Z = \int_{-\infty}^{\infty} f(x+iy)\psi(x)dx,$$

and that, moreover,

$$\int_{-\infty}^{\infty} f(x+iy)\psi(x)dx = \int_{-\infty}^{\infty} f(x+iy+iy_0)\psi(x+iy_0)dx$$

for any $y_0 \in C$, since both f and ψ are holomorphic in T^C. Thus the limit for $y \to 0, y \in C$ equals

$$\lim_{y\to 0, y\in C} \int_{-\infty}^{\infty} f(x+iy)\psi(x)dx = \int_{-\infty}^{\infty} f(x+iy_0)\psi(x+iy_0)dx,$$

so that it exists in Z' and is independent of the path. We see that it has no sense to speak of "boundary values in Z'".

Next we consider for an $y \in \text{pr } C'$ the function $f_0(\lambda;x,y) = f(x+\lambda y)$, $\lambda = \sigma + i\tau$, which is holomorphic in λ for $\tau > 0$. When we integrate $\alpha+1$ times with respect to λ:

$$f_{\alpha+1}(\lambda;x,y) = \int_{i\eta}^{\lambda} \int_{i\eta}^{\lambda_\alpha} \cdots \int_{i\eta}^{\lambda_1} f_0(\lambda_0;x,y)d\lambda_0 d\lambda_1 \ldots d\lambda_\alpha$$

with η a positive number, we get a function $f_{\alpha+1}$ holomorphic in λ for $\tau > 0$, that satisfies

$$|f_{\alpha+1}(\lambda;x,y)| \leq M(1+\|x\|+|\sigma|)^{\beta}\ (1+|\sigma|)^{\alpha+1}$$

for all $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$ and $0 < \tau \leq R$ for some $R > 0$ (see [11]) and

$$f_0(\lambda;x,y) = f(x+\sigma y+i\tau y) = \frac{d^{\alpha+1} f_{\alpha+1}(\sigma+i\tau;x,y)}{d\sigma^{\alpha+1}} \quad .$$

Let $\phi \in (S_{\alpha+1}^{\beta+n+\frac{1}{2}})_x$ and $\chi \in (\mathcal{D}([-1,1]))_\sigma$ with $\int \chi(\sigma)d\sigma = 1$, then

$$\sup_{\substack{0\leq p\leq\alpha+1\\ x\in\mathbb{R}^n,-1\leq\sigma\leq 1}} (1+\|x\|+|\sigma|)^{\beta+n+\frac{1}{2}} \left| \frac{d^p}{d\sigma^p} \phi(x+\sigma y)\chi(\sigma) \right| \leq B\|\phi\|_{\alpha+1}^{\beta+n+\frac{1}{2}}$$

for some $B > 0$. Finally

$$\left|\int_{-\infty}^{\infty} f(x+i\tau y)\phi(x)dx\right| = \left|\int_{-1}^{1}\int_{-\infty}^{\infty} f(x-\sigma y+\sigma y+i\tau y)\phi(x)\chi(\sigma)dx\ d\sigma\right| =$$

$$= \left|\int_{-1}^{1}\int_{-\infty}^{\infty} f(x+\sigma y+i\tau y)\phi(x+\sigma y)\chi(\sigma)\ dx\ d\sigma\right| =$$

$$= \left|\int_{-1}^{1}\int_{-\infty}^{\infty} f_{\alpha+1}(\lambda;x,y)\frac{d^{\alpha+1}}{d\sigma^{\alpha+1}}\phi(x+\sigma y)\chi(\sigma)dx\ d\sigma\right| \leq$$

$$\leq K\ M\ B\ \|\phi\|_{\alpha+1}^{\beta+n+\frac{1}{2}}$$

for some constant K and all τ with $0 \leq \tau \leq R$, so that the set

$$\{f(x+i\tau y)\}_{\substack{y\in \text{pr } C'\\ 0<\tau\leq R}}$$

is bounded in $(S_{\alpha+1}^{\beta+n+\frac{1}{2}})'$. Since Z is dense in $S_{\alpha+1}^{\beta+n+\frac{1}{2}}$ $f(x+iy)$ has a weak limit for $y \to 0$, $y \in C'$ in $(S_{\alpha+1}^{\beta+n+\frac{1}{2}})'$ according to E.3. By Schauder's theorem (E.1) and by C.4 the embedding map from $(S_{\alpha+1}^{\beta+n+\frac{1}{2}})'$ into $(S_{\alpha+2}^{\beta+n+1})'$ is compact and therefore maps weakly convergent sequences into strongly convergent sequences (E.2). So we have found that

(6.1) $\quad f^* \in (S^{\beta+n+1}_{\alpha+1})'$

and that f(x+iy) converges to f^* in the space $(S^{\beta+n+1}_{\alpha+2})'$. □

Remark 6.1. This result is somewhat stronger than that given in [1].

Remark 6.2. The correspondence between f and g in lemma 6.1 is 1-1, because, if $\alpha(\xi)$ is a C^∞-function with support in $O(a+1;C)$ and $\alpha(\xi) \equiv 1$ for $\xi \in \bar{O}(a;C)$, then for $\phi \in S$ the set $\{\psi \mid \psi(\xi) = \alpha(\xi)e^{-y\cdot\xi}\phi(\xi), y \in C_k, \|y\| \leq 1\}$ is a bounded set in S and $\alpha(\xi)e^{-y\cdot\xi}\phi(\xi) \to \alpha(\xi)\phi(\xi)$ in S as $y \to 0, y \in C_k$. Therefore $e^{-y\cdot\xi}g_\xi = \alpha(\xi)e^{-y\cdot\xi}g_\xi \to \alpha(\xi)g_\xi = g_\xi$ in S' as $y \to 0, y \in C_k$ with $g \in S'$ having its support in $\bar{O}(a;C)$ and

$$(6.2) \qquad f^* = \lim_{y\to 0, y\in C_k} f(x+iy) = \lim_{y\to 0, y\in C_k} \mathcal{F}[e^{-y\cdot\xi}g_\xi] = \mathcal{F}[g] \text{ in } S'.$$

Remark 6.3. The property of lemma 6.2 depends on the behaviour of f for $\|y\|$ small, while the condition on the support of $\mathcal{F}[f]$ in lemma 6.1 depends on the behaviour of f for $\|y\|$ large.

Next we give a topology to the class $H_1(a;C)$. Let a sequence $C_k \subset\subset C$ be given as in A.4 and let, moreover, an increasing sequence of continuous convex functions a_m on pr C be given with for m=0,1,2,...

$$a_m(y) < a_{m+1}(y) < a(y) \qquad \text{and} \lim_{m\to\infty} a_m(y) = a(y)$$

for all $y \in$ pr C, where a also is a continuous convex function on pr C. In some cases we need one more condition on the functions a_m. This condition will depend on a non-negative constant β and it yields the following property, called N_β:

(N_β) for any $\xi \in \partial\,\bar{O}(a_m;C)$ the distance d_ξ from ξ to $\partial O(a;C)$ satisfies

$$(6.3)\qquad d_\xi \geq \frac{\varepsilon_m}{||\xi||^\beta} \quad \text{when } ||\xi|| > 1 \quad \text{and}$$

$$d_\xi \geq \varepsilon_m \quad \text{when } ||\xi|| \leq 1 \quad ,$$

where $0 < \varepsilon_{m+1} < \varepsilon_m \leq \frac{1}{2}$ for $m=0,1,2,...$ and $\lim_m \varepsilon_m = 0$.

It is clear that N_β implies N_α if $\beta < \alpha$. The condition imposed on the functions a_m (let us call it N_β too) gives a bound for the velocity with which $a(y)-a_m(y)$ tends to zero as y approaches the boundary of pr C, if it tends to zero at all, what is possible only if $\beta > 0$. For $\beta = 0$ $a(y)-a_m(y) \geq \varepsilon_m$ for all $y \in$ pr C.

Let the positive weight functions M_m on T^C be

$$M_m(z) = \frac{e^{-\tilde{a}_m(y)}}{(1+||z||)^m(1+||y||^{-m})}$$

for m=0,1,2,... We define the Frêchet space

$$H^m(a_m;C) \overset{\text{def}}{=\!=} \operatorname{proj}\lim_{k\to\infty} A_\infty(M_m;T^{C_k})$$

(see B.4 and F.2); its elements are functions holomorphic in T^C. If we denote the closure of $H^m(a_m;C)$ in the space $A_\infty(M_m;\mathbb{R}^n+iC_k)$ by $H^m(a_m;C)^{\bar{k}}$, then $H^m(a_m;C) = \operatorname{proj}\lim_k H^m(a_m;C)^{\bar{k}}$ too in virtue of F.7. Finally we define

$$\tilde{H}(a;c) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{m\to\infty} H^m(a_m;C).$$

Using C.7 we find that the identity map from $A_\infty(M_m;\mathbb{R}^n+iC_k)$ into $A_\infty(M_{m+1};\mathbb{R}^n+iC_{k-1})$ is compact. Therefore the identity map from $H^m(a_m;C)$ into $H^{m+1}(a_{m+1};C)$ maps bounded sets into relatively compact sets.

(6.4) In fact this is true for the map from $H^m(a_m;C)$ into $H^{m+1}(a_m;C)$.

We will prove in lemma 6.5 that there is a q such that the identity map maps the neighborhood of zero $U = \{f \mid f \in H^m(a_m;C), \|f\|^m_{\bar{k}} \leq K\}$ in $H^m(a_m;C)$, thus also the unit ball in $H^m(a_m;C)^{\bar{k}}$, into a bounded set of $H^{m+q}(a_m;C)$. Hence the identity map from $H^m(a_m;C)$ into $H^{m+q+1}(a_m;C)$ is compact, thus also the map from $H^m(a_m;C)$ into $H^{m+q+1}(a_{m+q+1};C)$. Therefore $\tilde{H}(a;C)$ is an LS-space and it is a bornological, regular and complete Montel space. Also it follows that we can write it as an inductive limit of Banach spaces (compare F.9)

$$\tilde{H}(a;C) = \operatorname{ind}\lim_{m\to\infty} H^m(a_m;C)^{\bar{m}} .$$

Moreover, according to G.8 $\tilde{H}(a;C)$ is nuclear.

Like in remark 3.2

$$\tilde{H}(a;C) = \operatorname{ind}\lim_{l\to\infty} \operatorname{ind}\lim_{m\to\infty} H^m(a_l;C)^{\bar{m}} , \qquad \text{or}$$

(6.5)

$$\tilde{H}(a;C) = \operatorname{ind}\lim_{l\to\infty} \operatorname{ind}\lim_{m\to\infty} H^m(a_l;C) \quad ,$$

where the inductive limit for $m \to \infty$ yields an LS-space and the inductive limit for $l \to \infty$ is a strict inductive limit of complete spaces, as will follow from (6.4) and (6.6).

Next we consider the space $\tilde{S}'(a;C)$ of distributions in S' with support in one of the sets $\bar{O}(a_m;C)$ contained in the set $O(a;C)$, $m=0,1,2,\ldots$, namely

$$\tilde{S}'(a;C) \stackrel{\text{def}}{=\!=} \operatorname{ind}\lim_{m\to\infty}(S_m(a_m;C))' ,$$

where $(S_m(a_m;C))'$ is the space consisting of the m-th order distributions in S' with support in $\bar{O}(a_m;C)$. This last space is the strong dual of

$$S_m(a_m;C) \stackrel{\text{def}}{=\!=} W^m_{\infty,0}((1+\|\xi\|)^m, \bar{O}(a_m;C))$$

in virtue of the remark in D.2. From D.3 it follows that the restriction map from $S_{m+1}(a_{m+1};C)$ into $S_m(a_m;C)$ is compact, so that according to E.1 $\tilde{S}'(a;C)$ is an LS-space. Therefore $\tilde{S}'(a;C)$ is the strong dual of the FSspace

$$S(a;C) \overset{\text{def}}{=\!=} \operatorname{proj}\lim_{m\to\infty} S_m(a_m;C).$$

As before we can write $\tilde{S}'(a;C)$ as

$$\tilde{S}'(a;C) = \operatorname{ind}\lim_{l\to\infty} \operatorname{ind}\lim_{m\to\infty} (S_m(a_l;C))'. \tag{6.6}$$

Indeed, by F.4 (inductive limit with respect to l) and F.5 (with respect to m) there is a continuous map from ind $\lim_l S'(a_l;C)$ into $\tilde{S}'(a;C)$, where $S'(a_l;C)$ is defined as

$$S'(a_l;C) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{m\to\infty} (S_m(a_l;C))',$$

and the inverse of this map is continuous according to F.5. In virtue of F.12 and G.4 $S'(a_l;C)$ is an LS-space and in virtue of G.5 $S'(a_l;C)$ is a closed linear subspace of S' (it is the subspace of S' of distributions with support in the closed set $\bar{O}(a_l;C)$). Therefore the inductive limit for $l \to \infty$ is a strict inductive limit of LS-spaces. According to G.6 $S'(a_l;C)$ is nuclear and according to G.9 $\tilde{S}'(a;C)$ is nuclear too. Note, that $\tilde{S}'(a;C)$ is a strict inductive limit of closed subspaces of S', but that its topology is finer then the one induced by S'.

<u>Lemma 6.3</u>. *For any* $z \in T^C$, $e^{iz\cdot\xi} \in S(a;C)_\xi$.

<u>Proof</u>. We show that, for any m, $\|e^{iz\cdot\xi}\|_m$ is finite for every $z \in T^C$ by estimating the norm $\|e^{iz\cdot\xi}\|_{m,l}^p$ for $y \in C_k$. For $y \in C_k$ and $\xi \in C_{k+1}^*$ we have $\cos(\tilde{y},\tilde{\xi}) \geq \delta_k$ (see A.3 and A.4 for notations and symbols) so that (see also [11])

$$-\, y\cdot\xi \leq \|y\|\,\|\xi\|\,\delta_k.$$

For all $\xi_0 \in (C_{k+1}^*)^c$ there is a $y_0 \in \operatorname{pr} C_{k+2}$ with $\cos(y_0,\tilde{\xi}_0) \leq -\,\delta_{k+1}$; thus $-\,y_0\cdot\xi_0 \geq \|\xi_0\|\,\delta_{k+1}$ (see figure 6.1)

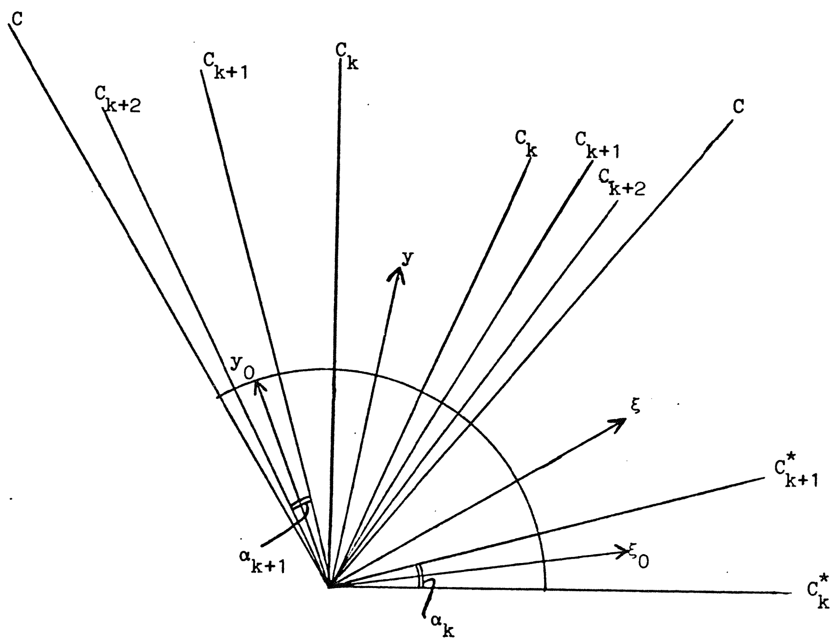

Figure 6.1

Then for every $\xi_0 \in (C^*_{k+1})^c \cap \bar{O}(a_1;C)$ it follows that

$$\|\xi_0\|\delta_{k+1} \le \sup_{y_0 \in \mathrm{pr}\, C_{k+2}} -y_0\cdot\xi_0 \le \sup_{y_0\in \mathrm{pr}\, C_{k+2}} a_1(y_0) \le \sup_{y_0\in \mathrm{pr}\, C_{k+2}} a(y_0) \overset{\mathrm{def}}{=\!=} b_k;$$

thus with $d_k = b_k/\delta_{k+1}$: $\|\xi_0\| \le d_k$. Now we have for $y \in C_k$

$$(6.7) \qquad \|e^{iz\cdot\xi}\|^p_{m,1} = \sup_{\substack{\xi\in O(a_1;C)\\ |q|\le m}} (1+\|\xi\|)^p \left| \frac{\partial^q}{\partial\xi^q} e^{iz\cdot\xi} \right| \le$$

$$\le \underset{\substack{\xi\in C^*_{k+1}\\ |q|\le m}}{} (1+\|\xi\|)^p \, \|z\|^{|q|} e^{-\|y\|\,\|\xi\|^{\delta_k}} +$$

$$+ \sup_{|q|\le m} (1+d_k)^p \, \|z\|^{|q|} e^{\tilde{a}_1(y)} .$$

Next we consider the function $(1+\rho)^p e^{-\|y\|\delta_\rho}$ for $\rho \ge 0$. The maximum is attained when $(1+\rho)^{p-1}e^{-\|y\|\delta_\rho}(p-(1+\rho)\|y\|\delta) = 0$, thus for

$$\rho = \frac{p-\|y\|\delta}{\|y\|\delta} = \frac{p}{\|y\|\delta} - 1$$

if this ≥ 0, otherwise for $\rho = 0$. The maximum is

$$\left(\frac{p}{\|y\|\delta}\right)^p e^{\|y\|\delta-p} \leq \left(\frac{p}{\delta}\right)^p \frac{1}{\|y\|^p}$$

if $p - \|y\|\delta \geq 0$, otherwise the maximum is 1. Finally we get for all $y \in C_k$

$$\|e^{iz\cdot\xi}\|^p_{m,1} \leq K_{p,k}(1+\|z\|)^m(1+\|y\|^{-p})e^{\tilde{a}_1(y)}. \quad \square$$

As (3.2) the following lemma holds:

Lemma 6.4. *For* $g \in \tilde{S}'(a;C)$ *and* $y \in C$

$$F[g_\xi e^{-y\cdot\xi}](x) = \langle g_\xi, e^{iz\cdot\xi}\rangle.$$

Proof. According to (6.6) we may assume that $g \in S'(a_1;C)$. Let $\alpha(\xi)$ be a C^∞-function with support in $O(a_1+1;C)$ and $\alpha(\xi) \equiv 1$ for $\xi \in \bar{O}(a_1;C)$. Then $\alpha(\xi)e^{-y\cdot\xi} \in S$ for each $y \in C$ and for all $\phi \in S$ we have

$$\langle F[g_\xi e^{-y\cdot\xi}],\phi\rangle = \langle g_\xi e^{-y\cdot\xi}, \int_{-\infty}^{\infty} e^{ix\cdot\xi}\phi(x)dx\rangle =$$

$$= \langle g_\xi \alpha(\xi)e^{-y\cdot\xi}, \int_{-\infty}^{\infty} e^{ix\cdot\xi}\phi(x)dx\rangle =$$

$$= \langle g_\xi, \int_{-\infty}^{\infty} \alpha(\xi)e^{iz\cdot\xi}\phi(x)dx\rangle_\xi.$$

Furthermore, $\alpha(\xi)e^{iz\cdot\xi}\phi(x) \in S_{\xi,x}$ and considering g_ξ as a distribution in $S'_{\xi,x}$ we get according to [8,(51.7)]

$$<F[g_\xi e^{-y\cdot\xi}],\phi>_x = <g_\xi,\alpha(\xi)e^{iz\cdot\xi}\phi(x)>_{\xi,x} =$$

$$= \int_{-\infty}^{\infty} <g_\xi,\alpha(\xi)e^{iz\cdot\xi}>_\xi \phi(x)dx =$$

$$= < <g_\xi,e^{iz\cdot\xi}>_\xi,\phi>_x;$$

thus $F[g_\xi e^{-y\cdot\xi}](x) = <g_\xi,e^{iz\cdot\xi}>$. □

Now we formulate the main theorem of this section:

Theorem 6.1. *The map* F: $\tilde{S}'(a;C) \to \tilde{H}(a;C)$ given by $F(g)(z) = <g_\xi,e^{iz\cdot\xi}>$ *for* $g \in \tilde{S}'(a;C)$ *is an isomorphism.*

Proof. The bijectivity of F follows from lemma 6.1 and remark 6.2.

Next we will prove the continuity of F. Again we use that $\tilde{S}'(a;C)$ and $\tilde{H}(a;C)$, being LS-spaces, are bornological and regular (see F.11, F.15 and F.16). Let $B \subset \tilde{S}'(a;C)$ be a bounded set. This means that a positive integer m and a constant M exist such that $B \subset (S_m(a_m;C))'$ and $\|g\|_{-m} \le M$ for all $g \in B$. According to lemma 6.4 and (6.8) the images $f = F(g)$ satisfy

$$|f(z)| = |<g_\xi,e^{iz\cdot\xi}>| \le \|g\|_{-m} \|e^{iz\cdot\xi}\| \le$$

$$\le MK_{m,k}(1+\|z\|)^m(1+\|y\|^{-m})e^{\tilde{a}_m(y)}$$

with $y \in C_k$. Thus $f \in H^m(a_m;C)$ and for any k $\|f\|_k^m \le MK_{m,k}$; this means that $F(B)$ is a bounded set in $\tilde{H}(a;C)$. Therefore F is continuous.

In order to prove the continuity of F^{-1} we consider a neighborhood of zero U in $H^m(a_m;C)$ with m arbitrary, that is to say there is a constant K and an integer k such that

$$U = \{f \mid f \in H^m(a_m;C), \|f\|_k^m \le K\}.$$

We will prove that the image $F^{-1}(U)$ of U is bounded in some $(S_l(a_m;C))'$ with $l > m$, from which it follows that F^{-1} is a continuous map from

$H^m(a_m;C)$ into $(S_l(a_m;C))'$. Since this is true for all m=0,1,2,... F^{-1} is a continuous map from $\tilde{H}(a;C)$ into $\tilde{S}'(a;C)$ according to F.5.

Let us assume more generally

$$f \in H^{m,p}(a_m;C) \stackrel{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} A_\infty\left(\frac{e^{-\tilde{a}_m(y)}}{(1+\|z\|)^m(1+\|y\|^{-p})}\,;\,T^{C_k}\right), \text{ and}$$

$$|f(z)| \leq K(1+\|z\|)^m(1+\|y\|^{-p})\, e^{\tilde{a}_m(y)} \qquad \text{for } y \in C_k.$$

According to lemma 6.1 the image under F^{-1} of such functions f consists of distributions in S' with support in $\bar{O}(a_m;C)$, so that it is contained in the linear subspace $S'(a_m;C)$ of S'. Since by lemma 6.1, lemma 6.4 and (6.2) the image g under F^{-1} of f is $g = F^{-1}[f^*]$, where $F[\cdot]$ denotes the Fourier transform in S', we get for all $\phi \in S$ taking into account (6.1)

$$(6.9) \qquad |\langle g,\phi\rangle| = |\langle f^*, F^{-1}[\phi]\rangle| = \|f^*\|^{-(m+n+1)}_{-(p+1)} \|F^{-1}[\phi]\|^{m+n+1}_{p+1} \leq$$

$$\leq Mc_r \|\phi\|^{p+n+1+r}_{m+n+1}$$

for any r > 0 and some positive constant M (see also H.1). Hence $F^{-1}(U)$ is bounded in $(S_l)'$ with $l = \max(m+n+1,p+n+2)$ and being contained in the linear subspace $S'(a_m;C)$ of S', it is bounded in $(S_l(a_m;C))'$. □

Remark 6.4. In fact we can show with the aid of (6.8) that

(6.10) F^{-1} is a continuous map from $H^m(a_m;C)^{\bar{k}}$ into $(S^{m+n+1+r}_{m+n+1}(a_m;C))'$ for any r > 0, and

(6.11) F is a continuous map from $(S^p_m(a_m;C))'$ into $H^{m,p}(a_m;C)$.

We conclude this section with the following lemma (see also remark 8.1):

Lemma 6.5. *The identity map from* $H^m(a_m;C)$ *into* $H^{m+n+3}(a_m;C)$ *is compact.*

Proof. As we have seen in the proof of theorem 6.1, a neighborhood of zero U in $H^m(a_m;C)$ is mapped by F^{-1} into a bounded set of $(S_{m+q}(a_m;C))'$

with $q = n+2$. According to (6.11) a bounded set in $(S_{m+q}(a_m;C))'$ is mapped by F into a bounded set of $H^{m+q}(a_m;C)$; according to (6.4) the identity map maps this bounded set into a relatively compact set of $H^{m+q+1}(a_m;C)$. □

7. NEWTON SERIES FOR NON-ENTIRE FUNCTIONS WITH DISTRIBUTIONAL BOUNDARY VALUES

In this section we derive the Newton series (1.1) for functions in $\tilde{H}(a;C)$, where the condition N_β holds for some $\beta \geq 0$.

Let a cone $C \subset \mathbb{R}^n$ and a convex function a on pr C be given. We suppose that h is a vector in C and that for any $y \in$ pr C, whenever $h = \|h\| y$,

$$\|h\| \leq \frac{\log 2}{a(y)} \qquad \text{if } a(y) > 0$$

or

$$\|h\| \text{ arbitrarily large} \qquad \text{if } a(y) \leq 0.$$

Furthermore, there is a positive number α depending on y and h, where $y \in C_k$ and $h \in C_l$ with $l \geq k$, such that for $s \in \mathbb{C}$ with $\text{Re } s \geq -\alpha$

$$y + (\text{Re } s)h \in C_l.$$

If Re $s \geq 0$ always $y + (\text{Re } s)h \in C$ for all y and h in C. We remind of the definition of the functions $\phi_N(\xi)$ in (4.5).

<u>Lemma 7.1</u>. *If* $y \in C_k$, $h \in C_l$ *and* Re $s \geq -\alpha$ *as above, the sequence* $e^{iz\cdot\xi}\phi_N(\xi)$ *tends for* $N \to \infty$ *to* $e^{iz\cdot\xi - sh\cdot\xi}$ *in* $S(a;C)$, *when in* $S(a;C)$ *the property* N_β *(6.3) holds for some* $\beta \geq 0$.

<u>Proof</u>. According to lemma 6.3

$$e^{iz\cdot\xi - kh\cdot\xi} = e^{i(x+i(y+kh))\cdot\xi} \qquad \text{for } k=0,1,2,\ldots$$

and

$$e^{iz\cdot\xi - sh\cdot\xi} = e^{i(x-\text{Im } sh+i(y+\text{Re } sh))\cdot\xi}$$

belong to $S(a;C)$. Also for $\xi \in O(a;C)$ $\phi_N(\xi)$ tends to $e^{-sh\cdot\xi}$ as $N \to \infty$. We show that the set $\{e^{iz\cdot\xi}\phi_N(\xi)\}_{N=0}^{\infty}$ is bounded in $S(a;C)$, from which the lemma follows by means of G.2. Using (5.3) we get for all $y \in C_k$

$$\| e^{iz\cdot\xi}\phi_N(\xi)\|_m \le C \sup_{\substack{\xi\in\bar{O}(a_m;C)\\ |q|\le m}} C'(s,q,\alpha)(1+\|\xi\|)^{m+\alpha\beta+\beta m} e^{\alpha h\cdot\xi}\left|\frac{\partial^q}{\partial\xi^q}e^{iz\cdot\xi}\right| \le$$

$$\le C(s,m,\alpha,1)(1+\|z\|)^m(1+\|y-\alpha h\|^{-(m+\alpha\beta+\beta m)})e^{\tilde{a}_m(y-\alpha h)}$$

as in (6.8), which is independent of N. For $\alpha = 0$ the norm $\|\cdot\|_m$ is bounded by

$$C(s,m,k,\beta)(1+\|z\|)^m(1+\|y\|^{-(m+\beta m+1+r)})e^{\tilde{a}_m(y)}$$

with $r > 0$. □

Let f be an element of $\tilde{H}(a;C)$, where in $\tilde{H}(a;C)$ the functions a_m satisfy the condition N_β for some $\beta \ge 0$ and let $g \in \tilde{S}'(a;C)$ be its image under F^{-1}. As in (4.4), using theorem 6.1 and lemma 7.1 we derive the Newton series for $y \in C_k, h \in C_l\,(l\ge k)$ and $\operatorname{Re} s \ge -\alpha$

$$(7.1)\qquad f(z+ish) = \langle g_\xi, e^{iz\cdot\xi-sh\cdot\xi}\rangle = \langle g_\xi, \lim_{N\to\infty} e^{iz\cdot\xi}\phi_N(\xi)\rangle =$$

$$= \lim_{N\to\infty}\sum_{k=0}^{N}\binom{s}{k}\langle g_\xi, e^{iz\cdot\xi}(e^{-h\cdot\xi}-1)^k\rangle = \sum_{k=0}^{\infty}\binom{s}{k}\Delta_{ih}^k f(z),$$

where $\alpha > 0$ depends on y and h such that $y - \alpha h \in C_l$. In order to describe the topology in which the series converges it is convenient to change the complex variable $z = x + iy$ into $z = x + i(y-\alpha h)$, so that we get

$$(7.2)\qquad f(z+i(s+\alpha)h) = \sum_{k=0}^{\infty}\binom{s}{k}\Delta_{ih}^k f(z+i\alpha h),$$

which is now valid for all $y \in C, h \in C$ and $\operatorname{Re} s + \alpha \ge 0$, $\alpha \ge 0$ arbitrary.

According to (5.3) $e^{-\alpha h\cdot\xi}\phi_N(\xi)$ is a multiplier in S' and also in

$\tilde{S}'(a;C)$. Therefore the sequence $\{g_\xi e^{-\alpha h\cdot\xi}\phi_N(\xi)\}_{N=1}^{\infty}$ converges weakly in $\tilde{S}'(a;C)$ and since this is a Montel space, it converges strongly too (see F.10). Hence the series (7.2) converges in the topology of $\tilde{H}(a;C)$. This yields the convergence of the series (7.1), namely in one of the norms of the topology of $\tilde{H}(a;C)$. Thus (7.1) and (7.2) certainly converge uniformly in z on compact subsets of T^C.

Let us consider more precisely the norms in which (7.2) converges. Suppose α_l are C^∞-functions identical 1 for $\|\xi\| \le l$ and 0 for $\|\xi\| \ge l+1$, $l=1,2,\ldots$ Then if $\{g_N\}_{N=1}^{\infty}$ is a bounded set in $(S_m^p(a_m;C))'$, $\alpha_l g_N$ tends to g_N as $l \to \infty$ uniformly in N in the space $(S_m^{p+r}(a_m;C))'$ for any $r > 0$, since for $\phi \in S(a;C)$

$$|\langle g_N-\alpha_l g_N,\phi\rangle| = |\langle g_N,(1-\alpha_n)\phi\rangle| \le$$

$$\le \|g_N\|_{-m}^{-p} \sup_{\substack{\xi\in\bar{O}(a_m;C)\\ |q|\le m}} \{(1+\|\xi\|)^{p+r}\left|\frac{\partial^q}{\partial\xi^q}(1-\alpha_l)\phi\right|\frac{1}{(1+\|\xi\|)^r}\}\le$$

$$\le K\|\phi\|_m^{p+r}\frac{1}{l^r} < \epsilon\ \|\phi\|_m^{p+r},$$

for l sufficiently large. Let us take $g_N = g_\xi e^{-\alpha h\cdot\xi}\phi_N(\xi)$, so that according to (5.3) the set $\{g_N\}$ is bounded in $(S_m^t(a_m;C))'$ with $t = p + \alpha\beta + \beta m$ if $\alpha > 0$ or $t = p + 1 + r + \beta m$ if $\alpha = 0$, when $g \in (S_m^p(a_m;C))'$. We know already from section 4 that $\alpha_l g_N$ tends to $\alpha_l g_\infty \overset{\text{def}}{=\!=} \alpha_l(\xi)g_\xi e^{-(s+\alpha)h\cdot\xi}$ as $N \to \infty$ strongly in $(\mathcal{E}_m(\{\xi \mid \|\xi\| \le l+1\}))'$ for every l. So we get finally for $\phi \in S(a;C)$ and $\varepsilon > 0$

$$|\langle g_N-g_\infty,\phi\rangle| \le |\langle g_N-\alpha_l g_N,\phi\rangle| + |\langle\alpha_l g_N-\alpha_l g_\infty,\phi\rangle| + |\langle\alpha_l g_\infty-g_\infty,\phi\rangle| \le$$

$$\le \frac{1}{3}\varepsilon\|\phi\|_{m,m}^{t+r} + \frac{1}{3}\varepsilon\ \|\phi\|_{m,m}^{0} + \frac{1}{3}\varepsilon\ \|\phi\|_{m,m}^{t+r} \le \varepsilon\ \|\phi\|_{m,m}^{t+r},$$

when we first choose $l = l(\varepsilon)$ so large, that the first and the third term on the right-hand side of the first line can be estimated and when we next take $N \ge N_0(l(\varepsilon),\varepsilon)$ so that the middle term can be estimated. Therefore

$g_\xi e^{-\alpha h\cdot\xi}\phi_N(\xi)$ converges for $N\to\infty$ strongly in $(S_m^{t+r}(a_m;C))'$. If $f\in\tilde{H}(a;C)$ satisfies

$$(7.3)\qquad |f(z)|\le M_k(1+\|z\|)^m(1+\|y\|^{-p})e^{\tilde{a}_m(y)}\qquad\text{for } y\in C_k,\ k=1,2,\ldots,$$

then in the above where $g=F^{-1}(f)$ we take $p+n+1+r$ instead of p and $m+n+1$ instead of m according to (6.9). In that case t is determined by choosing r so small that the smallest integer larger than $p+n+1+(\alpha+m+n+1)\beta+2r$ equals

$$(7.4)\qquad [p+n+1+(\alpha+m+n+1)\beta]+1=t$$

or if $\alpha=0$, the smallest integer larger than $p+n+2+(m+n+1)\beta+3r$ must be equal to

$$(7.5)\qquad [p+n+2+(m+n+1)\beta]+1=t.$$

In virtue of (6.11) and (5.4) the Newton series (7.2) for functions f satisfying (7.3) valid for $y\in C, h\in C$ and $\operatorname{Re} s\ge-\alpha$ with $\alpha\ge0$ converges according to:

$$\forall\varepsilon>0,\ \forall l,\ \exists N_0(\varepsilon,l)\ge N_1(s),\ \forall z\in T^{C_l}\text{ and }\forall N\ge N_0$$

$$(7.6)\quad |f(z+i(s+\alpha)h)-\sum_{k=0}^{N}\binom{s}{k}\Delta_{ih}^{k}f(z+i\alpha h)|<\varepsilon A(s)(1+\|z\|)^{m+n+1}(1+\|y\|^{-t})e^{\tilde{a}_m(y)},$$

where $N_1(s)$ is determined by (5.1), $A(s)$ by (5.4) and t by (7.4) if $\alpha>0$ or by (7.5) if $\alpha=0$. Replacing $z+i\alpha h$ by z in (7.6) we see that the Newton series (7.1), which is valid for $y\in C_k, h\in C_l$ and $\operatorname{Re} s\ge-\alpha$ with $y-\alpha h\in C_l$, converges according to

$$(7.7)\qquad \forall\varepsilon>0,\ \exists N_0(\varepsilon)\ge N_1(s),\ \forall z\in T^{C_k}\text{ and }\forall N\ge N_0$$

$$|f(z+ish)-\sum_{k=0}^{N}\binom{s}{k}\Delta_{ih}^{k}f(z)|<\varepsilon A(s)(1+\|z\|)^{m+n+1}(1+\|y-\alpha h\|^{-t})e^{\tilde{a}_m(y-\alpha h)},$$

where t is given by (7.4) and again $A(s) = 1/|\Gamma(-s)|$. As mentioned in section 5.I there is uniform convergence in s on compact subsets of $\{s \mid s \in \mathbb{C}, \text{Re } s \geq -\alpha\}$. Note that since we used (5.4), ε depends on α in (7.6).

If Re $s \geq 0$, y may tend to zero within C_l in (7.1). Hence in virtue of lemma 6.2 and the fact that t is independent of l in (7.7), the Newton series (7.1) is valid in S' for $y = 0$. As we have seen in the derivation of (5.3), the series $\phi_N(\xi)$ and its derivatives converge absolutely. Therefore the Newton series (7.1) and (7.2) are absolutely convergent.

Finally we restate the results in

Theorem 7.1. *Let* $h \in C$ *with* $\|h\| \leq \log 2/a(\tilde{h})$ *if* $a(\tilde{h}) > 0$ *or* $\|h\|$ *arbitrary if* $a(h) \leq 0$ *and let* f *be an element of* $\tilde{H}(a;C)$, *when the condition* N_β *holds for some* $\beta \geq 0$. *If* $\alpha > 0$ *is such that* $y - \alpha h \in C_l$ *for some* $y \in C$ *and* l *such that* $h \in C_l$, *the Newton series (7.1) is valid for this* y *and* h, *when* Re $s \geq -\alpha$. *The series (7.1) converges absolutely in one of the norms of* $\tilde{H}(a;C)$ *or more precisely it converges according to (7.7), when* f *satisfies (7.3); moreover, (7.1) is valid in* S' *for* $y \in C_l \cup \{0\}$ *with* l *arbitrary, when* Re $s \geq 0$. *When* Re $s \geq -\alpha$ *with* $\alpha \geq 0$ *arbitrary, the Newton series (7.2) holds for all* $y \in C$ *and* $h \subset C$; *then the series (7.2) converges absolutely in the topology of* $\tilde{H}(a;C)$ *or more precisely it converges according to (7.6), when* f *satisfies (7.3). In both cases (7.1) and (7.2) converge univormly in* s *on compact subsets of* $\{s \mid s \in \mathbb{C}, \text{Re } s \geq -\alpha\}$.

8. FUNCTIONS HOLOMORPHIC IN TUBULAR RADIAL DOMAINS NOT HAVING BOUNDARY VALUES

We have considered holomorphic functions of the complex variable $z = x + iy$ that, regarded as distributions in x, belong to S' for each $y \in C$ just as its distributional boundary value, i.e. the limit as y tends to zero within a compact subcone C_k. Now we will consider holomorphic functions that belong to S' only for $y \in C$; it turns out that they also belong to Z' just as their limit as y tends to zero within C.

First we investigate when a function f, holomorphic in T^B with B some open convex set in $\mathbb{R}^n$, belongs to S' or Z' for each $y \in B$. We consider f as an element of Z' (see H.4) under the definition:

$$(8.1) \qquad \forall\psi \in Z: \quad \langle f,\psi\rangle_Z \overset{\text{def}}{=\!=} \int_{-\infty}^{\infty} f(x+iy)\psi(x)dx \overset{\text{not}}{=\!=} \langle f(z),\psi(x)\rangle .$$

The space $Z(\mathbb{R}^n)$ of the restrictions to $\mathbb{R}^n$ of functions in Z can be isomorphically embedded into S^0, the space of rapidly decreasing continuous functions in $\mathbb{R}^n$ with the topology induced by the norms $\|\cdot\|_0^p$, p=0,1,2,... (see H.1). So any continuous linear functional on $Z(\mathbb{R}^n)$ can be extended to a continuous linear functional on S^0 in a unique way, since $Z(\mathbb{R}^n)$ is dense in S^0. Furthermore $(S^0)' \subset S'$, so that any element of $Z(\mathbb{R}^n)'$ belongs to S'. Definition (8.1) means that f regarded as element of Z' is an element of $Z(\mathbb{R}^n)'$ and thus this implies that f is an element of S' too for each $y \in B$. In lemma 8.2 we will see that conversely $f \in S'$ for each $y \in B$ implies $f \in Z'$ under definition (8.1).

In view of the use we will make in part II of lemma 8.1, f is considered there as an element of $\mathcal{E}_x$ for each $y \in B$ and there are no restrictions on the growth of f. Taking into account theorem 3.1 we see that $\mathcal{F}$ transforms the dual of $\tilde{H} = \tilde{H}(\mathbb{R}^n)$ into $\mathcal{E}$ and that $\mathcal{F}^{-1}$ transforms $\mathcal{E}$ into the dual of $\tilde{H}$.

Lemma 8.1. *Let f be a function holomorphic in T^B with B some open convex set in $\mathbb{R}^n$ and let $y_0 \in B$. Then for all y with $y + y_0 \in B$*

$$f(x+iy+iy_0) = \mathcal{F}(e^{-y\cdot\zeta} g(y_0)_\zeta)\ (x)$$

with $g(y_0) = \mathcal{F}^{-1}(f(x+iy_0)) \in \tilde{H}'$. Furthermore $\{f(x+iy)\}_{y\in K}$ is a bounded set in $\mathcal{E}_x$ for any $K \subset\subset B$.

Proof. We have $\sup\{|f(x+iy)| \mid x \in S, y \in K\} \le C(S,K)$ for any compact set $S \subset\subset \mathbb{R}^n$. Since f is holomorphic, also the derivatives of f are uniformly bounded on $S \times K$ by $C_1(\varepsilon)\cdot C(S_\varepsilon,K_\varepsilon)$ where S_ε and K_ε are ε-neighborhoods of S and K such that $K_\varepsilon \subset\subset B$ and where $C_1(\varepsilon)$ is a constant depending on ε. Therefore the second statement follows:

$$\sup_{y\in K} \| f(x+iy)\|_m^{E} \le C_m \qquad \text{for } m=0,1,2,\ldots$$

Let $\rho < 1$ and choose ε so small that $y + y_0 \in B$ whenever

$$|y_i| \le \eta \stackrel{\text{def}}{=\!=} \frac{n\varepsilon}{\rho} \qquad \text{for } i=1,\ldots,n.$$

Note that also $y + y_0 \in B$ when $|y_i| \le \varepsilon$, $i=1,\ldots,m$. For such an y and for a $\phi \in E'_x$ we have

$$<f(x+iy+iy_0),\phi_x>_E = <\sum_{k=0}^{\infty} \frac{(iy\cdot D)^k}{k!} f(x+iy_0),\phi_x>.$$

In order to show that the series converges in E_x we use G.2. Therefore we first estimate

$$|D^l f(x+iy_0)| \le \frac{l!}{(2\pi)^n} \int_{|z_1|=\eta} \cdots \int_{|z_n|=\eta} \left|\frac{f(x+iy_0+z)}{z^{l+\bar{1}}}\right| dz \le$$

$$\le l!\, \frac{1}{\eta^{\bar{l}}} \sup_{|z_i|\le\eta} |f(x+iy_0+z)| \;.$$

Hence

$$\sup_{\substack{x\in S\\ |\alpha|\le m}} |D^\alpha \sum_{k=0}^{N} \frac{(iy\cdot D)^k}{k!} f(x+iy_0)| \le \sup_{\substack{x\in S\\ |z_i|\le\eta}} |f(x+iy_0+z)| \sup_{|\alpha|\le m} \frac{\alpha!}{\eta^{\bar{\alpha}}} \sum_{k=0}^{N} (y\cdot \frac{\bar{1}}{\eta})^k \le$$

$$\le C \sum_{k=0}^{N} \rho^k \le \frac{C}{1-\rho} \quad ,$$

which is independent of N. With $\psi(\zeta) = F(\phi)(\zeta) \in \tilde{H}$ we now can write

$$<f(x+iy+iy_0),\phi_x>_E = \sum_{k=0}^{\infty} <f(x+iy_0), \frac{(-iy\cdot D)^k}{k!} \phi_x> =$$

$$= \sum_{k=0}^{\infty} <g(y_0)_\zeta, \frac{(-y\cdot\zeta)^k}{k!} \psi(\zeta)>_H = \sum_{k=0}^{\infty} <\frac{(-y\cdot\zeta)^k}{k!} g(y_0)_\zeta \,, \psi(\zeta)> =$$

$$= \lim_{N\to\infty} <\sum_{k=0}^{N} \frac{(-y\cdot\zeta)^k}{k!} g(y_0)_\zeta,\psi(\zeta)> = <g(y_0+y)_\zeta, \psi(\zeta)> \;.$$

Thus the weak limit for $N \to \infty$ exists in $\tilde{H}'$ and since $\tilde{H}$ is a Montel space, the strong limit exists and equals $e^{-y\cdot\zeta}g(y_0)_\zeta$. Hence

$$f(x+iy+iy_0) = F(e^{-y\cdot\zeta}g(y_0)_\zeta)(x)$$

for y with $|y_i| \le \epsilon$; by analytical continuation this formula holds for every y with $y + y_0 \in B$. □

<u>Lemma 8.2</u>. *A function* f, *holomorphic in* T^B, *is an element of* S'_x *for each* $y \in B$ *if and only if for each compact set* $K \subset\subset B$ *there are positive constants* M(K) *and* m(K) *such that*

$$(8.2) \qquad |f(z)| \le M(K)(1+\|x\|)^{m(K)} \qquad \text{for } z \in T^K.$$

In this case f belongs to Z' too for each $y \in B$ under definition (8.1).

<u>Proof</u>. It is clear that a function f satisfying (8.2) belongs to S' for each $y \in B$. Take $y_0 \in B$, then $g(y_0)_\xi = F^{-1}[f(x+iy_0)] \in S'_\xi$. Since $Z(\mathbb{R}^n)$ is contained in the space of restrictions to $\mathbb{R}^n$ of functions in $\tilde{H}$ and since $Z(\mathbb{R}^n)$ is dense in S, lemma 8.1 holds with $g(y_0)_\xi \in S'_\xi$. Hence

$$F^{-1}[f(x+iy+iy_0)]_\xi = e^{-y\cdot\xi}g(y_0)_\xi \in S'_\xi$$

for all y with $y + y_0 \in B$. According to [11,26.2] the set

$$\{e^{-y\cdot\xi}g(y_0)_\xi\}_{y \in K_1}$$

is bounded in S'_ξ, whenever $y_0 + K_1 = K$ is a compact set in B. Therefore there are constants M(K), m(K) and $\alpha = \alpha(K)$ such that the α-th primitive F of f:

$$F(z) \underline{\underline{\text{def}}} \int_0^x \int_0^{t_\alpha} \cdots \int_0^{t_1} f(t_0+iy)dt_0\, dt_1 \cdots dt_\alpha$$

satisfies $|F(z)| \le M(K)(1+\|x\|)^{m(K)}$ for all $x \in \mathbb{R}^n$ and $y \in K$. For any $K \subset\subset B$ there is an ε-neighborhood K_ε of K also compact in B.

Using Cauchy's formula we find

$$|f(z)| = |D^{\bar{\alpha}}F(z)| \leq C_\varepsilon \sup_{\substack{x\in\mathbb{R}^n \\ y\in K_\varepsilon}} |F(z)| \leq C_\varepsilon M(K_\varepsilon)(1+\|x\|)^{m(K_\varepsilon)}$$

which is condition (8.2). □

From (8.2) it follows that for any $y_0 \in B$ and $\psi \in Z$

$$<f(z),\psi(x)> = <f(z+iy_0),\psi(x+iy_0)> .$$

Therefore the limit f^* as $y \to 0, y \in C$ of $f(x+iy)$ exists in Z', namely for all $\psi \in Z$

$$<f^*,\psi> = \lim_{\substack{y\to 0 \\ y\in C}} <f(x+iy),\psi(x)> = \lim_{\substack{y\to 0 \\ y\in C}} <f(x+iy+iy_0),\psi(x+iy_0)> =$$

$$= <f(x+iy_0),\psi(x+iy_0)> .$$

This limit f^* is independent of the path $y \to 0$ in C. Its inverse Fourier transform $g = \mathcal{F}^{-1}(f^*)$ is an element of $\mathcal{D}'$ and with $\phi = \mathcal{F}[\psi] \in \mathcal{D}'$ we get

$$<g,\phi>_{\mathcal{D}} = <f^*,\psi>_Z = <f(x+iy),\psi(x+iy)>_Z = <g(y)_\xi, e^{y\cdot\xi}\phi(\xi)>_{\mathcal{D}} =$$

$$= <e^{y\cdot\xi}g(y)_\xi,\phi(\xi)>_{\mathcal{D}}.$$

Hence

$$g_\xi = e^{y\cdot\xi}g(y)_\xi$$

is independent of $y \in B$. Finally $g(y)_\xi = e^{-y\cdot\xi}g_\xi$ belongs to S'_ξ, so that

(8.3) $f(z) = \mathcal{F}[e^{-y\cdot\xi}g_\xi](x)$ with $g \in \mathcal{D}'$ and $e^{-y\cdot\xi}g_\xi \in S'$ for all $y \in B$,

(compare corollary 5.2 and [11,26.2]). Remark that the limit $f^* \in Z'$ no longer obeys definition (8.1).

We now assume that B is an open convex cone C in $\mathbb{R}^n$ and that f(z) is of exponential growth a for $\|y\|$ large. As before we can show that $g \in \mathcal{D}'$ has its support contained in $\bar{O}(a;C)$, since we only need the growth of $|f(z)|$ for $\|y\|$ large and not for $\|y\|$ small (see remark 6.3 and the proof of lemma 6.1 in [11]). Let a be a continuous convex function on pr C and let $\{C_k\}_{k=1}^{\infty}$ be as in A.3. So we have

Theorem 8.1. *The following statements are equivalent*

(1) f *holomorphic in* T^C, *has the property that for each* k *there are positive numbers* M(k) *and* m(k) *such that*

$$(8.4) \qquad |f(z)| \le M(k)(1+\|z\|)^{m(k)}\, e^{\tilde{a}(y)} \quad \textit{for all } y \in C_k \textit{ with } \|y\| \ge 1/k$$

(2) $f(z) = \mathcal{F}[e^{-y\cdot\xi} g_\xi](x)$, *where* g_ξ *is a distribution in* $\mathcal{D}'$ *with* $e^{-y\cdot\xi} g_\xi \in S'$ *for* $y \in C$ *and with its support contained in* $\bar{O}(a;C)$.

The difference with lemma 6.1 is that there f(x+iy) attains a boundary value in S' for $y \to 0, y \in C_k$ and here only in Z', while in both cases f(x+iy) belongs to S'_x for $y \in C$.

The next lemma shows that there are no cases "in between":

Lemma 8.3. *If the limit in* Z' *for* $y \to 0, y \in C$ *of a holomorphic function* f *satisfying (8.4) belongs to* S', *then* f(x+iy) *attains this limit also in* S' *for* $y \to 0, y \in C_k$ *and* f *satisfies the stronger condition*

$$(8.5) \qquad |f(z)| \le M(k)(1+\|z\|)^m (1+\|y\|^{-m})\, e^{\tilde{a}(y)}, \qquad z \in \mathbb{R}^n + iC_k,$$

for all k *and some* m *independent of* k.

Proof. In virtue of (8.3) we have $e^{-y\cdot\xi} g_\xi \in S'$ for all $y \in C \cup \{0\}$. Let us fix a vector $y_0 \in$ pr C. As in the proof of lemma (8.2) the set

$$\{e^{-\tau y_0\cdot\xi} g_\xi\}_{0\le\tau\le1}$$

is bounded in S' (see [11,26.2]). Indeed,

$$e^{-\tau y_0\cdot\xi} g_\xi = \frac{e^{-\tau y_0\cdot\xi}}{1+e^{-y_0\cdot\xi}} g_\xi + \frac{e^{-\tau y_0\cdot\xi}}{1+e^{-y_0\cdot\xi}} g_\xi e^{-y_0\cdot\xi}$$

and one can show that the function

$$\frac{e^{-\tau y_0\cdot\xi}}{1+e^{-y_0\cdot\xi}}$$

and its derivatives with respect to ξ are uniformly bounded for $0 \le \tau \le 1$. Using E.3 and the fact that the limit for $\tau \to 0$ exists in $\mathcal{D}'$, we find that, according to (8.3), $f(x+iy)$ attains its limit for $y \to 0, y \in C_k$ in S'. Since the support of g is contained in $\bar{O}(a;C)$ by theorem 8.1, g belongs to $S'(a;C)$ (see section 6) and in virtue of lemma 6.4 and (6.8) we get

$$|f(z)| = |\mathcal{F}[e^{-y\cdot\xi}g_\xi](x)| = |\langle g_\xi, e^{iz\cdot\xi}\rangle| \le \|g\|_{-m} \| e^{iz\cdot\xi}\|_m \le$$

$$\le M(k)(1+\|z\|)^m(1+\|y\|^{-m})e^{\tilde{a}(y)}$$

for all $y \in C_k$ and some non-negative m. □

Indeed there are no spaces "in between". For, if f satisfies (8.4) and if moreover for some $y_0 \in \mathrm{pr}\, C$

$$(8.6) \qquad |f(x+i\tau y_0)| \le M(1+\|x\|)^l \tau^{-l}, \qquad 0 < \tau \le 1,$$

then its boundary value f^* belongs to S' according to the proof of lemma 6.2. Now it follows from lemma 8.3 that f, has to satisfy (8.5) for some $m > l$.

Remark 8.1. If f satisfies (8.4) and (8.6), then f must satisfy (8.5) for all $m > l$, for, if not (thus if for some $m_0 > l$ f does not satisfy (8.5)), then, for a sufficiently large p, f^p would not satisfy (8.5) with $m = pm_0 > pl + n + 2$, which contradicts (6.10) and (6.11). Thus $\overline{H^l(a_1;C)^k}$ can be embedded continuously into $\overline{H^m(a_1;C)^{k+1}}$ for all $m > l$. It follows that the identity map transforms a neighborhood of zero in $H^l(a_1;C)$ into a bounded set of $H^m(a;C)$, $m > l$, so that already the identity map from $H^l(a_l;C)$ into $H^{l+1}(a_l;C)$ is compact (compare lemma 6.5).

Remark 8.2. If f, satisfying (8.4) and (8.6), would satisfy (8.5) for $m = 1$, then $H^l(a_l;C)^{\bar{k}}$ would be embedded continuously *onto* $H^l(a_l;C)^{\overline{k+1}}$ and $H^l(a_l;C) = H^l(a_l;C)^{\bar{k}}$ for k=1,2,..., would be a Banach space according to [8, corollary 3 of theorem 17.1].

9. TOPOLOGICAL SPACES OF HOLOMORPHIC FUNCTIONS NOT HAVING BOUNDARY VALUES AND THEIR FOURIER TRANSFORMS

In this section we will give a topology to the space of holomorphic functions satisfying (8.4) and to the space of their Fourier transforms, such that Fourier transformation is an isomorphism.

We first give a topology to the space of holomorphic functions f satisfying (8.4). It is quite obvious how we must proceed. Let

$$T_p^{C_k} = T^{C_k} \cap \{z \mid z = x + iy \in \mathbb{C}^n,\ \|y\| > \frac{1}{p}\}$$

and let $\{a_l\}_{l=1}^{\infty}$ be an increasing sequence of continuous convex functions on pr C converging to a as before (section 6). We define the spaces

$$H^{m*}(a_l;C_k,p) \overset{\text{def}}{=\!=} A_\infty\left(\frac{e^{-\tilde{a}_l(y)}}{(1+\|z\|)^m},\ T_p^{C_k}\right),$$

$$H^*(a_l;C) \overset{\text{def}}{=\!=} \operatorname{proj}\lim_{k\to\infty}\ \operatorname{ind}\lim_{m\to\infty} H^{m*}(a_l;C_k,k)$$

and finally

$$\tilde{H}^*(a;C) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{l\to\infty} H^*(a_l;C).$$

We will find that $H^*(a_l;C)$ is a projective limit of nuclear LS-spaces and that $\tilde{H}^*(a;C)$ is a strict inductive limit. However, to see this it is necessary to give an equivalent definition of $H^*(a_l;C)$, which is less clear, but which enables us to derive the above mentioned structure.

Denote for any k by $\{C_{k+1/m}\}_{m=1}^{\infty}$ a decreasing sequence of convex relatively compact subcones of C_{k+1} with

$$\overline{\text{pr } C_k} \subset \text{pr } C_{k+\frac{1}{m+1}} \subset \overline{\text{pr } C_{k+\frac{1}{m+1}}} \subset \text{pr } C_{k+\frac{1}{m}} \subset \overline{\text{pr } C_{k+\frac{1}{m}}} \subset \text{pr } C_{k+1}$$

for m=1,2,... and with

$$\bigcap_{m=1}^{\infty} \text{pr } C_{k+\frac{1}{m}} = \overline{\text{pr } C_k}.$$

From C.3 it follows that the following identity maps are continuous

$$H^{m*}(a_l;C_{k+1},k+1) \to H^{m*}(a_l;C_{k+1/m},k+1/m) \to H^{m*}(a_l;C_k,k),$$

so that according to F.5 and F.6 also

$$H^*(a_l;C) = \text{proj}\lim_{k\to\infty} \text{ind}\lim_{m\to\infty} H^{m*}(a_l;C_{k+1/m},k+1/m).$$

But now the identity map from $H^{m*}(a_l;C_{k+1/m},k+1/m)$ into $H^{m+1*}(a_l;C_{k+1/(m+1)},k+1/(m+1))$ is compact according to C.7. Also the conditions HS_1 and HS_2 of G.7 are satisfied. Hence $H^*(a_l;C)$ is the projective limit of nuclear LS-spaces and it is itself nuclear and complete (see G.9 and F.2). Furthermore the bounded sets of $H^*(a_l;C)$ are relatively compact, since they are for each k bounded in the Montel space $\text{ind}\lim_m H^{m*}(a_l;C_{k+1/m},k+1/m)$, thus relatively compact there. Moreover, a bounded set in $H^*(a_l;C)$, being bounded in $\text{ind}\lim_m H^{m*}(a_l;C_{k+1/m},k+1/m)$ for each k, is for all k bounded in some $H^{m*}(a_l;C_{k+1/m},k+1/m)$ where $m = m(k)$ depends on k and therefore bounded in $H^{m(k)*}(a_l;C_k,k)$ for all k. Finally $\tilde{H}^*(a;C)$ is nuclear and complete (see G.9 and F.14), since the inductive limit for $l \to \infty$ is strict, as we will see in the sequel (remark 9.2).

It is less obvious how to topologize the space of inverse Fourier transforms g of f^*, where f satisfies (8.4). We know already that $g \in \mathcal{D}'$, $e^{-y\cdot\xi}g_\xi \in S'$ for $y \in C$ and that the support of g is contained in $O(a;C)$ (see theorem 8.1). We will first investigate some more properties of such distributions g. Taking into account (8.3) and (8.4) we find that the set

$$\{e^{-y\cdot\xi}g_\xi\}_{\substack{y\in C_k \\ 1/k\le \|y\| \le 1}}$$

is bounded in S'. Thus there are constants K'_k and $m(k)$ and functions $G'_{\alpha,y}, |\alpha| \le m(k)$ (see B.5), such that

$$g(y)_\xi = e^{-y\cdot\xi}g_\xi = \sum_{|\alpha|\le m(k)} D^\alpha G'_{\alpha,y}(\xi)$$

with $G'_{\alpha,y}$ continuous functions (the second order primitive of a bounded measure is a continuous function), which satisfy

$$|G'_{\alpha,y}(\xi)| \le K'_k(1+\|\xi\|)^{m(k)}, \; y \in C_k, \; \frac{1}{k} \le \|y\| \le 1.$$

Furthermore, $e^{y\cdot\xi}g(y)_\xi = g_\xi \in \mathcal{D}'$ is independent of y, thus for each k there are constant K_k and $m(k)$ and functions $G_\alpha, |\alpha| \le m(k)$, such that

$$g = \sum_{|\alpha|\le m(k)} D^\alpha G_\alpha$$

where G are continuous functions with support in an ε-neighborhood of $O(a;C)$ satisfying $|G_\alpha(\xi)| \le K_k(1+\|\xi\|)^{m(k)} e^{y\cdot\xi}$ for all $y \in C_k$ with $1/k \le \|y\| \le 1$, so that

$$(9.1) \qquad |G_\alpha(\xi)| \le K_k(1+\|\xi\|)^{m(k)} \inf_{\substack{y\in C_k \\ 1/k\le\|y\|\le 1}} e^{y\cdot\xi}.$$

Now we are able to describe the space of distributions g, but before doing this we give simpler weight functions defining the same topology as the weight function

$$M_k^1(\xi) \stackrel{def}{=\!=} \inf_{\substack{y\in C_k \\ 1/k\le \|y\| \le 1}} e^{y\cdot\xi}$$

We remind of the numbers δ_k defined in A.4. We can take the compact subcones C_k of C so, that the sequence δ_k is decreasing. Thus we have $1 > \delta_k > \delta_{k+1} > 0$ for all $k=1,2,\ldots$ and $\lim_k \delta_k = 0$. Since $y\cdot\xi = \|y\|\,\|\xi\|\cos(\tilde{y},\tilde{\xi})$, we can find for each k an integer $p > k$, such that for all $\xi \in C^*_{p+1}$

$$\text{(9.2)} \qquad \delta_k \frac{1}{k} \|\xi\| \ge \frac{1}{p} \|\xi\| \ge \inf_{\substack{y\in C_p \\ 1/p\le \|y\| \le 1}} y\cdot\xi \ge \delta_p \frac{1}{p} \|\xi\| .$$

We define two more weight functions, which will give the same space of distributions g, namely

$$M_k^2(\xi) \stackrel{def}{=\!=} e^{\frac{1}{k}\|\xi\|} \quad \text{and} \quad M_k^3(\xi) \stackrel{def}{=\!=} e^{\delta_k \frac{1}{k}\|\xi\|} .$$

We have seen in (9.1) that, for all k,g is an element of the strong dual of

$$S^{k*}(a_1;C)_i \stackrel{def}{=\!=} \operatorname{proj}\lim_{m\to\infty} S_m^{k*}(a_1;C)_i$$

with

$$S_m^{k*}(a_1;C)_i \stackrel{def}{=\!=} W^m_{\infty,0}((1+\|\xi\|)^m M_k^i(\xi);\ \bar{O}(a_1;C)) \qquad (i=1,2,3),$$

when we take $i = 1$. Hence g belongs to

$$S^*(a_1;C)' \stackrel{def}{=\!=} \operatorname{proj}\lim_{k\to\infty} (S^{k*}(a_1;C)_i)'$$

which is independent of i in virtue of F.6. Indeed, for all k there is a $p > k$ such that the following maps are continuous

$$S^{k*}(a_1;C)_3 \to S^{p*}(a_1;C)_2 \to S^{p*}(a_1;C)_1 \to S^{p*}(a_1;C)_3$$

according to F.5, (9.2) and the fact that the set $(C^*_{p+1})^c \cap \bar{O}(a_l;C)$ is compact. For i=2, $S^*(a_l;C)'$ is defined in the simplest way, but we prefer i=3 and in the sequel we delete the subscript 3, so that

$$S^{k*}(a_l;C) \overset{\text{def}}{=\!=\!=} S^{k*}(a_l;C)_3 .$$

Finally we define

$$\tilde{S}^*(a;C)' \overset{\text{def}}{=\!=\!=} \operatorname*{ind\,lim}_{l\to\infty} S^*(a_l;C)' .$$

According to G.4 $S^{k*}(a_l;C)$ is an $F\bar{S}$-space and according to G.5 and G.6 $(S^{k*}(a_l;C))'$ is a nuclear LS-space. Hence $S^*(a_l;C)'$ is the projective limit of nuclear LS-spaces and therefore it is itself nuclear and complete. Furthermore, according to G.5 $(S^{k*}(a_l;C))'$ is a closed linear subspace of $(S^{k*}(\mathbb{R}^n))'$; hence the projective limit $S^*(a_l;C)'$ is a closed linear subspace of $\text{proj lim}_k\ (S^{k*}(\mathbb{R}^n))'$, so that $S^*(a_l;C)'$ is a closed linear subspace of $S^*(a_{l+1};C)'$. Therefore the inductive limit for $l \to \infty$ is a strict inductive limit of complete spaces and it follows from G.9 and F.14 that $\tilde{S}^*(a;C)'$ is nuclear and complete.

Like lemma 6.3 we have here too:

Lemma 9.1. *For any* $z = x + iy$, *with* $y \in C_k$ *and* $\|y\| \geq 1/k$,

$$e^{iz\cdot\xi} \in S^{k*}(a_l;C)_\xi .$$

Proof. For all m we estimate the norms of $e^{iz\cdot\xi}$, when $y \in C_k$ with $\|y\| \geq 1/k + \varepsilon$

$$(9.3)\quad \|e^{iz\cdot\xi}\|^k_{m,l} \overset{\text{def}}{=\!=\!=} \sup_{\substack{\xi\in\bar{O}(a_l;C)\\ |p|\leq m}} (1+\|\xi\|)^m e^{\delta_k \frac{1}{k}\|\xi\|} \left|\frac{\partial^p}{\partial\xi^p} e^{iz\cdot\xi}\right| \leq \text{as in (6.7)}$$

$$\leq \sup_{\substack{\xi\in C^*_{k+1}\\ |p|\leq m}} (1+\|\xi\|)^m \|z\|^p e^{-(\|y\| - \frac{1}{k})\delta_k\|\xi\|} +$$

$$+ \sup_{|p|\leq m} (1+d_k)^m \|z\|^p e^{\delta_k \frac{1}{k} d_k + \tilde{a}_l(y)} \leq$$

$$\leq C(m,k,\varepsilon)(1+\|z\|)^m e^{\tilde{a}_1(y)}$$

according to (6.8). □

In particular (9.3) holds for $y \in C_{k-1+1/m}$ with $\|y\| \geq 1/(k-1+1/m)$. It is clear that in lemma 9.1 also $e^{iz\cdot\xi} \in S^{r*}(a_1;C)_\xi$ for all $r \geq k$. Therefore it has sense to formulate the following lemma.

Lemma 9.2. *For any* $y \in C$ *and* $g \in S^*(a_1;C)'$

$$F[e^{-y\cdot\xi}g_\xi](x) = \langle g_\xi, e^{iz\cdot\xi}\rangle .$$

Proof. We know that for all $y_0 \in C$ and all $g \in S^*(a_1;C)'$

$$e^{-y_0\cdot\xi} g_\xi \in S'. \tag{9.4}$$

Indeed, let us take $y_0 \in C_k$, then in virtue of the fact that for $\xi \in C^*_{k+1}$

$$-y_0\cdot\xi \leq - \|y_0\|\,\delta_k\|\xi\| \leq -\delta_p \frac{1}{p}\|\xi\|$$

for p large enough and the fact that $(C^*_{k+1})^c \cap \bar{O}(a_1;C)$ is compact, multiplication by $\exp(-y_0\cdot\xi)$ is a continuous map from $S(a_1;C)$ into $S^{r*}(a_1;C)$ for all $r \geq p$. Hence multiplication by $\exp(-y_0\cdot\xi)$ is continuous from $S^*(a_1;C)'$ into $S'(a_1;C)$. Thus

$$e^{-y_0\cdot\xi} g_\xi = g(y_0)_\xi \in S'(a_1;C).$$

Let us choose an $y \in C$, say $y \in C_k$. Then according to lemma 9.1 $e^{iz\cdot\xi} \in S^{k*}(a_1;C)$. Take $y_0 = \delta_k / (k\|y\|)\, y$, so that $\|y_0\| = \delta_k/k$. Then multiplication by $\exp(y_0\cdot\xi)$ is a continuous map from $S^{k*}(a_1;C)$ into $S(a_1;C)$. In particular

$$e^{y_0\cdot\xi+iz\cdot\xi} = e^{i(z-iy_0)\cdot\xi} \in S(a_1;C).$$

Now we can apply lemma 6.4

$$F[e^{-y\cdot\xi}g_\xi](x) = F[e^{-y_0\cdot\xi}g_\xi\ e^{y_0\cdot\xi-y\cdot\xi}](x) = \langle g(y_0)_\xi, e^{i(z-iy_0)\cdot\xi}\rangle =$$

$$= \langle e^{y_0\cdot\xi}g(y_0)_\xi, e^{iz\cdot\xi}\rangle = \langle g_\xi, e^{iz\cdot\xi}\rangle. \qquad \square$$

Like theorem 6.1 we have

Theorem 9.1. *The map* $F\colon \tilde{S}^*(a;C)' \to \tilde{H}^*(a;C)$ *given by* $F(g)(z) = \langle g_\xi, e^{iz\cdot\xi}\rangle$ *for* $g \in \tilde{S}^*(a;C)'$ *is an isomorphism.*

Proof. From theorem 8.1, (9.4), lemma 9.2 and the fact that Fourier transform in S' is 1-1, it follows that F in a 1-1 map from $S^*(a_l;C)'$ onto $H^*(a_l;C)$. In order to prove the continuity of F it is sufficient (see F.5) to show that, for all l and k, F is a bounded map from the bornological and regular space $(S^{k*}(a_l;C))'$ into ind $\lim_m H^{m*}(a_l;C_{k-1},k-1)$.

So let B be a bounded set in $(S^{k*}(a_l;C))'$, this means that for some m $B \subset (S^{k*}_m(a_l;C))'$ and, for all $g \in B$, $\|g\|^{-k*}_{-m} \le K$ for some $K > 0$. The images $f = F(g)$ satisfy according to (9.3)

$$|f(z)| = |\langle g_\xi, e^{iz\cdot\xi}\rangle| \le K\,\|e^{iz\cdot\xi}\|^{k*}_m \le K_{m,k}(1+\|z\|)^m e^{\tilde{a}_l(y)}$$

for $y \in C_k$ with $\|y\| \ge 1/(k-1)$. Thus $F(B)$ is a bounded set in $H^{m*}(a_l;C_{k-1},k-1)$ and therefore bounded in ind $\lim_m H^{m*}(a_l;C_{k-1},k-1)$.

Next we prove the continuity of F^{-1}. Again it would be sufficient to show that, for all l and p, F^{-1} is a bounded map between the LS-spaces ind $\lim_m H^{m*}(a_l;C_{k+1/m},k+1/m)$ and $(S^{p*}(a_l;C))'$ with $k \ge p$ depending on p. So we may start with a bounded set in $H^{m*}(a_l;C_{k+1/m},k+1/m)$ for some m, which is certainly bounded in $H^{m*}(a_l;C_k,k)$. Therefore let us start with a bounded set A in $H^{m*}(a_l;C_k,k)$, where k will be determined presently. Elements of A are holomorphic in T^Γ_k with $\Gamma = C_k$, so that we cannot expect that $F^{-1}(A) \subset (S^{p*}(a_l;C))'$.

We choose a positive integer p and we take k so large that

$$\frac{1}{k} < \delta_p \frac{1}{p}.$$

Let $y_0 \in C_k$ be such that

$\frac{1}{k} < \| y_0 \| < \delta_p \frac{1}{p}.$

Then $f(z+iy_0)$ is holomorphic in $\mathbb{R}^n + iC_k \cup \{0\}$ for $f \in A$ and it satisfies there

$$|f(z+iy_0)| \leq M(1+\|z\|)^m e^{\tilde{a}_1(y+y_0)} \leq M'(1+\|z\|)^m e^{\tilde{a}_1(y)},$$

since $\tilde{a}_1$ is homogeneous and convex: $\frac{1}{2}\tilde{a}_1(y+y_0) = \tilde{a}_1(\frac{1}{2}y+\frac{1}{2}y_0) \leq \frac{1}{2}\tilde{a}_1(y)+\frac{1}{2}\tilde{a}_1(y_0)$. Hence, according to lemma 6.1 and (6.2), the set $B' = \{g(y_0) | g(y_0) = \mathcal{F}^{-1}[f(x + iy_0)], f \in A\}$ is bounded in $(S^{n+1}_{m+n+1}(a_1;C_k))'$, thus also bounded in $(S_{m+n+1}(a_1;C_k))'$. Since $\| y_0 \| < \delta_p/p$, multiplication by $\exp(y_0 \cdot \xi)$ is a continuous map from $S^{p*}_m(a_1;C_k)$ into $S_m(a_1;C_k)$ for every m, so that the set

$$B = \{g \mid g_\xi = e^{y_0 \cdot \xi} g(y_0)_\xi, \; g(y_0) \in B'\}$$

is bounded in $(S^{p*}_{m+n+1}(a_1;C_k))'$.

Let y be such that $y + y_0 \in C_p$ and $\| y+y_0 \| > 1/p$, then as in lemma 9.1

$$e^{i(x+i(y+y_0))\cdot\xi} \in S^{p*}(a_1;C_k).$$

According to lemma 6.1 and lemma 6.4

$$f(x+iy+iy_0) = \mathcal{F}[e^{-y\cdot\xi} g(y_0)_\xi](x) = \langle g(y_0)_\xi, e^{iz\cdot\xi}\rangle =$$

$$= \langle g(y_0)_\xi, e^{y_0\cdot\xi} e^{i(x+i(y+y_0))\cdot\xi}\rangle =$$

$$= \langle e^{y_0\cdot\xi} g(y_0)_\xi, e^{i(x+i(y+y_0))\cdot\xi}\rangle ,$$

so that $\mathcal{F}(B) = A$. Hence $\mathcal{F}^{-1}(A)$ is bounded in $(S^{p*}_{m+n+1}(a_1;C_k))'$, thus $\mathcal{F}^{-1}$ is continuous from $H^*(a_1;C)$ into $(S^{p*}(a_1;C_k))'$ in virtue of the definition of the projective limit (F.1) and F.11. Ofcourse this is true for any larger k, so that $\mathcal{F}^{-1}$ is continuous from $H^*(a_1;C)$ into proj $\lim_k$ $(S^{p*}(a_1;C_k))'$. According to G.5 $(S^{p*}a_1;C_{k+1}))'$ is a closed linear subspace of

$(S^{p*}(a_l;C_k))'$; hence proj $\lim_k (S^{p*}(a_l;C_k))' = (S^{p*}(a_l;C))'$. So we have found that F^{-1} maps $H^*(a_l;C)$ continuously into $(S^{p*}(a_l;C))'$ for each $p=1,2,\ldots$, thus into $S^*(a_l;C)'$ according to F.4. □

Remark 9.1. In fact we have shown that

(9.5) $\mathcal{F}^{-1}$ maps a bounded set A of $H^*(a_l;C)$, that is a set bounded for each k in $H^{m(k)*}(a_l;C_k,k)$ with m(k) depending on k, into a bounded set B of $S^*(a_l;C)'$ that is bounded in $(S^{p*}_{m(q)+n+1}(a_l;C))'$ for each $p=1,2,\ldots$, $q = q(p) > p/\delta_p$,

and that

(9.6) $\mathcal{F}$ maps B into a bounded set of $H^*(a_l;C)$ that is bounded in $H^{m(q)+n+1*}(a_l;C_p,p-1)$ for each $p=1,2,\ldots$

Remark 9.2. We see that $S^*(a_l;C)'$ and $H^*(a_l;C)$ are isomorphic and since ind $\lim_l S^*(a_l;C)' = \tilde{S}^*(a;C)$ is a strict inductive limit, ind $\lim_l H^*(a_l;C) = \tilde{H}^*(a;C)$ is a strict inductive limit too.

10. NEWTON SERIES FOR NON-ENTIRE FUNCTIONS WITHOUT BOUNDARY VALUES

In this section we derive the Newton series (1.1) for functions in $\tilde{H}^*(a;C)$, when a more general condition than N_β holds.

Let a cone $C \subset \mathbb{R}^n$ and a convex function a on pr C be given. As in (6.3) the increasing sequence of convex functions a_l on pr C cannot be arbitrary, but there is a bound for the velocity with which $a(y) - a_l(y)$ tends to zero as y approaches the boundary of pr C. We assume this bound to be larger than in (6.3), namely the condition on the function a_l must yield the property

for any $\xi \in \partial\bar{O}(a_l;C)$ the distance d_ξ from ξ to $\partial O(a;C)$ satisfies

(10.1) $$d_\xi \geq \varepsilon_l K(\eta) e^{-\eta\|\xi\|} \qquad \text{for all } 1 \geq \eta > 0 \ (\xi \neq 0),$$

where $0 < \varepsilon_{l+1} < \varepsilon_l \leq \frac{1}{2}$ for $l=0,1,2,\ldots$ and $1 \geq K(\eta) > 0$.

If $K(\eta) \geq \rho > 0$ for all $1 \geq \eta > 0$, property (10.1) is identical to N_β (6.3) with $\beta = 0$; in all other cases, namely when $K(\eta)$ tends to zero as $\eta \to 0$, property (10.1) is more general than (6.3) for every $\beta > 0$. Instead of (5.3) we now get

$$(10.2) \qquad \forall \xi \in \Omega(\varepsilon(\xi)): \quad |D^k \phi_N(\xi)| \leq C(s,k,\alpha,\eta)\, e^{\eta \|\xi\|} e^{\alpha h \cdot \xi}$$

for all $1 \geq \eta > 0$ and $\alpha \geq 0$. A similar formula holds instead of (5.4).

As in section 7 we suppose that h is a vector in C and that for any $y \in \mathrm{pr}\, C$, whenever $h = \|h\| y$, $\|h\| \leq \log 2/a(y)$ in case $a(y) > 0$ or $\|h\|$ arbitrarily large in case $a(y) \leq 0$. Furthermore, there is a positive number α depending on y and h, where $y \in C_k$, $\|y\| > 1/k$ and $h \in C_r (r \geq k)$, such that for $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq -\alpha$

$$z + ish \in T_r^{C_r} .$$

If $\mathrm{Re}\, s \geq 0$ always $y + (\mathrm{Re}\, s)\, h \in C$ for all y and h in C.

Like lemma 7.1 we have

<u>Lemma 10.1</u>. *If* $y \in C_k$, $\|y\| > 1/k$, $h \in C_r$ *and* $\mathrm{Re}\, s \geq -\alpha$ *as above, the sequence* $e^{iz\cdot\xi}\phi_N(\xi)$ *tends for* $N \to \infty$ *to* $e^{iz\cdot\xi - sh\cdot\xi}$ *in* $S^{p*}(a_l;C)$ *for all* l *and* $p \geq r$, *when in* $S^{p*}(a_l;C)$ *the property (10.1) holds.*

<u>Proof</u>. According to lemma 9.1

$$e^{iz\cdot\xi - kh\cdot\xi} = e^{i(x+i(y+kh))\cdot\xi}$$

$$e^{iz\cdot\xi - sh\cdot\xi} = e^{i(x - \mathrm{Im}\, sh + i(y+\mathrm{Re}\, sh))\cdot\xi} \qquad \text{for } k=0,1,2,\ldots \text{ and}$$

belong to $S^{p*}(a_l;C)$ for all l and $p \geq r$. Furthermore, the sequence converges in each point $\xi \in O(a;C)$. We show that the set $\{e^{iz\cdot\xi}\phi_N(\xi)\}_{N=0}^{\infty}$ is bounded in $S^{p*}(a_l;C)$, from which the lemma follows by means of G.2. Choose an $y \in C_k$ with $\|y\| > 1/k$ and a $h \in C_r$ such that $\|y - \alpha h\| \geq 1/r + \varepsilon$; now choose $\eta < \varepsilon$. Using (10.2) we get for $p \geq r$

$$\| e^{iz\cdot\xi}\phi_N(\xi)\|^{p^*}_{m,1} \le$$

$$\le C \sup_{\xi\in\bar{O}(a_1;C)} C(s,q,\alpha,\eta)(1+\|\xi\|)^m e^{\eta\|\xi\| + \delta_p/p\|\xi\| + \alpha h\cdot\xi}\left|\frac{\partial^q}{\partial\xi^q} e^{iz\cdot\xi}\right| \le$$

$$\le C'(s,m,\varepsilon,\eta)(1+\|z\|)^m e^{\tilde{a}_1(y)}$$

as in (9.3). □

With this lemma we derive the Newton series like we did in (7.1) for functions f in $\tilde{H}^*(a;C)$, where the condition that yields property (10.1) holds

$$(10.3)\qquad f(z+ish) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta^k_{ih} f(z),$$

valid for all $y \in C_k$ with $\|y\| > 1/k$, $h \in C_r$ $(r\ge k)$ and Re $s \ge -\alpha$, where $\alpha > 0$ depends on y and h such that $y - \alpha h \in C_r$ and $\|y-\alpha h\| > 1/r$. In order to describe the convergence we prefer to write the series as (compare 7.2)

$$(10.4)\qquad f(z+i(s+\alpha)h) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta^k_{ih} f(z+i\alpha h),$$

valid for all $y \in C$ and $h \in C$, when Re $s \ge -\alpha$ with $\alpha \ge 0$ arbitrary.

According to (10.2) multiplication by $e^{-\alpha h\cdot\xi}\phi_N(\xi)$ is a continuous map for each r from $S^{r*}(a_1;C)$ into $S^{p*}(a_1;C)$ for all $p > r$. Therefore the sequence $\{g_\xi\ e^{-\alpha h\cdot\xi}\phi_N(\xi)\}_{N=1}^{\infty}$ converges weakly in $(S^{r*}(a_1;C))'$ for each r if $g \in S^*(a_1;C)'$ and since $S^{r*}(a_1;C)$ is a Montel space, it converges in the topology of $S^*(a_1;C)'$ (see F.10 and F.1). In virtue of theorem 9.1 the series (10.4) converges in the topology of $\tilde{H}^*(a;C)$. This yields the convergence of the series 10.3, namely in one of the norms of the topology of $\tilde{H}^*(a;C)$. Thus (10.3) and (10.4) certainly converge uniformly in z on compact subsets of T^C.

Let us consider more precisely the convergence of (10.3) and (10.4). Like in section 7 we can show that the sequence $\{g_\xi\ e^{-\alpha h\cdot\xi}\phi_N(\xi)\}_{N=0}^{\infty}$ with $g \in (S^{r*}_{m(r)}(a_1;C))'$ converges strongly in $(S^{p*}_{m(r)}(a_1;C))'$ for all $p > r$. Let f be holomorphic in T^C and satisfy for all k=1,2,...

$$(10.5) \qquad |f(z)| \le M_k(1+\|z\|)^{m(k)} e^{\tilde{a}_1(y)} \quad \text{for } y \in C_k \text{ with } \|y\| > 1/k.$$

Furthermore, let for each p=1,2,... k_p be such that

$$\frac{1}{k_p} < \delta_p \frac{1}{p} .$$

Then using (9.5) and (9.6) we find that the Newton series (10.4), valid for $y \in C$, $h \in C$ and Re $s \ge -\alpha$, $\alpha \ge 0$, converges according to

$$(10.6) \qquad \begin{aligned} &\forall p, \forall \varepsilon > 0,\ \exists N_0(\varepsilon,p) \ge N_1(s),\ \forall z \in T^{C_p}_{p-1} \text{ and } \forall N \ge N_0 \\ &\left|f(z+i(s+\alpha)h) - \sum_{k=0}^{N} \binom{s}{k} \Delta^k_{ih} f(z+i\alpha h)\right| < \\ &< \varepsilon A(s)(1+\|z\|)^{m(k_p)+n+1} e^{\tilde{a}_1(y)} , \end{aligned}$$

where $N_1(s)$ is determined by (5.1) and A(s) by (5.4). Similarly the series (10.3), valid for $y \in C_k$, $\|y\| > 1/k$, $h \in C_r$ and Re $s \ge -\alpha$ with $y - \alpha h \in C_r$ and $\|y-\alpha h\| > 1/r$, with f satisfying (10.5) for this k converges according to

$$(10.7) \qquad \begin{aligned} &\forall \varepsilon > 0,\ \exists N_0(\varepsilon) \ge N_1(s),\ \forall z \in T^{C_p}_{p-1} \text{ and } \forall N \ge N_0 \\ &\left|f(z+ish) - \sum_{r=0}^{N} \binom{s}{r} \Delta^r_{ih} f(z)\right| < \varepsilon A(s)(1+\|z\|)^{m(k)+n+1} e^{\tilde{a}_1(y-\alpha h)} \end{aligned}$$

with p such that $\delta_p/p > 1/k$.

We restate the results in

<u>Theorem 10.1</u>. *Let* $h \in C$ *with* $\|h\| \le \log 2/a(\tilde{h})$ *if* $a(\tilde{h}) > 0$ *or* $\|h\|$ *arbitrary if* $a(\tilde{h}) \le 0$ *and let* f *be an element of* $\tilde{H}^*(\dot{a};C)$, *where the condition that yields property (10.1) holds. If* $\alpha > 0$ *is such that* $y - \alpha h \in C_r$ *and* $\|y-\alpha h\| > 1/r$ *for some* $y \in C$ *and* r *such that* $h \in C_r$, *the Newton series (10.3) is valid for this* y *and* h, *when* Re $s \ge -\alpha$. *The series (10.3)*

converges absolutely in one of the norms of $\tilde{H}^*(a;C)$ *or, more precisely, it converges according to (10.7) when* f *satisfies (10.5) for* k *with* $y \in C_k$, $\|y\| > 1/k$. *When* $\mathrm{Re}\ s \geq -\alpha$ *with* $\alpha \geq 0$ *arbitrary, the Newton series (10.4) holds for all* $y \in C$ *and* $h \in C$; *then the series (10.4) converges absolutely in the topology of* $\tilde{H}^*(a;C)$ *or, more precisely, it converges according to (10.6) when* f *satisfies (10.5). In both cases (10.3) and (10.4) converge uniformly in* s *on compact subsets of* $\{s \mid s \in \mathbb{C}, s \geq -\alpha\}$.

Remark 10.1. With the same conditions as in theorem 7.1 or theorem 10.1 the formula

$$ih\cdot Df(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \Delta_{ih}^k f(z)$$

can be derived; this has been done in [3] for entire functions, where also other formulas, which need compact supports, are given similarly to theorem 4.1.

Remark 10.2. It seems unnecessary to take the strict inductive limit as a_1 tends to a in $\tilde{H}^*(a;C)$ (see remark 9.2), since all the lemma's and theorems have been derived for each a_1 seperately; also we could have avoided the strict inductive limits for $k \to \infty$ in (3.6) (remark 3.2) and for $l \to \infty$ in (6.5) and (6.6). However, in a subsequent paper these inductive limits will be no longer strict and they will be essential. There we do not require that the function f holomorphic in T^C is of polynomial growth in $\|x\|$, but it may be of exponential growth in $\|z\|$. In that case f(x+iy) belongs to $\mathcal{D}'_x$ for each $y \in C$ and f is the Fourier transform of an analytic functional in Z'. We will prove a theorem concerning functions f of exponential type, holomorphic in a tubular radial domain, and the sets carrying the analytic functionals of which f is the Fourier transform, similarly to lemma 6.1 and theorem 8.1 of this paper.

LITERATURE

[1] Boas, R.P. & Creighton Buck, R., *Polynomial expansions of analytic functions,* 2nd pr. Ergebnisse der Mathematik und ihre Grenzgebiete, left. 19, Springer-Verlag, Berlin, 1964.

[2] Floret, K. & Wloka, J., *Einführung in die Theorie der lokalkonvexen Räume,* Lecture Notes in Mathematics no. 56, Springer-Verlag, Berlin, 1968.

[3] Kioustelidis, J., *Eine einheitliche Methode zur Herleitung von Reihenentwicklungen für ganze Funktionen vom Exponentialtyp,* Compositio Math., 26 (1973) 201-232.

[4] Nörlund, N.E., *Vorlesungen über Differenzenrechnung,* Chelsea Publ. Co, N.Y., 1954.

[5] Rainville, E.D., *Special functions,* MacMillan Co., N.Y., 1960

[6] Schwartz, L., *Théorie des distributions, Tome I,* Hermann & Cie, Paris, 1951.

[7] Semadeni, Z., *Banach spaces of continuous functions I,* PWN, Warszawa, 1971.

[8] Treves, F., *Topological vector spaces, distributions and kernels,* Pure and applied mathematics, series no. 25, Academic Press, N.Y., 1967.

[9] Whitney, H., *Functions differentiable on the boundary of regions,* Ann. of Math., 35 (1934) 482-485.

[10] Whitney, H., *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc., 36 (1934) 63-89.

[11] Vladimirov, V.S., *Methods of the theory of functions of many complex variables* (translated from Russian), The M.I.T. Press, 1966.

[12] Wloka, J., *Grundräume und verallgemeinerte Funktionen,* Lecture Notes in Mathematics no. 82, Springer-Verlag, Berlin, 1969.